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# Skew-Product Dynamical Systems: Applications to Difference Equations

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## 1 Introduction

One of the earliest difference equations, the Fibonacci sequence, was introduced in 1202 in "Liberabaci," a book about the abacus, by the famous Italian Leonardo di Pisa, better known as Fibonacci. The problem may be stated as follows: how many pairs of rabbits will there be after one year when starting with one pair of mature rabbits, if each pair of rabbits give birth to a new pair each month starting when it reaches its maturity age of two months? If $F(n)$ is the number of pairs of rabbits at the end of $n$ months,

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then the difference equation that represents this model is given by the second order difference equation

$$F(n+1) = F(n) + F(n-1), \text{ with } F(0) = 1,\ F(1) = 1.$$

This very early connection between biology and difference equations has been, until recently, fairly dormant due largely to the creation of differential Calculus by Isaac Newton in 1732 and the dominance of differential equations in all natural sciences and economics.

Two great mathematicians have contributed significantly to the modern theory of difference equations; Henri Poincaré and George Birkhoff. In "Sur les equations linéaires aux differentielles ordinaires et aux différences finies" [61] Poincaré studied the asymptotic representation of solutions of nonautonomous linear difference equations. The Poincaré-Perron Theorem [20] is the extension of Poincaré Theorem by Oscar Perron in 1921 [58]. Walter Gautschi [31] showed how to use Poincaré–Perron Theorem to find minimal (recessive) solutions of linear nonautonomous difference equations. This has been a significant observation in the area of orthogonal polynomials and special functions since every sequence of a monic polynomial must satisfy a second order difference equations [8, 20, 73]. The importance of the existence of minimal solutions has been observed by Pincherle [60] much earlier in 1894, where he showed a continued fraction converges if and only if the associated second order difference equation has a minimal solution [20]. In a series of papers by George Birkhoff and his collaborators [6, 7], the formal theory of analytic difference equations has been established. Jet Wimp and Doran Zeilberger [74] applied Birkhoff Theory to compute the asymptotics of special functions and combinatorial identities. The study of the asymptotics of combinatorial identities was greatly facilitated by Zeilberger's algorithm by which one can produce a second order difference equation that represents a given combinatorial identity [74].

Another area where difference equations have played a prominent role is numerical analysis. Here one would approximate a given differential equation, through a discretization method, by a difference equation. However, these discretization methods may lead to instabilities and sometimes chaotic behavior. To remedy this situation, Ronald Mickens introduced a nonstandard discretization scheme which is dynamically consistent [2, 51, 55].

The modern theory of difference equations can be traced back to the 60's and 70's. In 1964, Alexander Nicoli Sharkovsky introduced his fundamental theorem on the periods of continuous maps on the real line [21, 69]. Part of Sharkovsky Theorem was later discovered in 1975, independently, by T.Y. Li and James Yorke in "Period three implies chaos" [50]. In addition to introducing "chaos" in mathematics, the Li–Yorke paper was instrumental in introducing Sharkovsky Theorem in English which made it accessible to scientists in the west. In 1978, Mitchell Feigenbaum [28] discovered a universal constant, the "Feigenbaum number," that is shared by unimodal continuous maps on the real line [21].

Since then, chaos theory has been in the frontier of science, particularly in physics and economics. In biology, however, progress in the use of difference equations as models has been rather slow. In 1976, difference equations received a huge boost through the fundamental work of Robert May who publicized the use of difference equations as biological models in his ever popular paper "Simple mathematical models with very complicated dynamics" [53]. The last twenty years witnessed an exponential growth in the area of mathematical biology with several new scientific journals and books devoted to the subject. But until recently, the favorite models for biologists have been the continuous ones, i.e., differential equations [19, 43, 59, 70].

Let us now move on to more recent developments. There are by now two schools in difference equations. The first school views difference equations as the discrete analogue of differential equations and analysis. It is no wonder, that the majority in this school came over from differential equations. Books representing this school are those of Lakschmikantham and Trigiante [48], Agarwal [1], Kelley and Peterson [37], Mickens [54], Kocic and Ladas [39], Kulenovic and Ladas [45], Sedaghat [65], Elaydi [20].

The second school views difference equations as iterations of maps or as discrete dynamical systems. The questions raised here are concerned with stability, bifurcation and chaos. Representing this school are the books by Devaney [16], Holmgren [34], Strogatz [71], Alligood et al [3], Martelli [52], Sandefur [64], Cushing [11, 12], Kulenovic and Merino [46], Elaydi [21]. The first two authors do not even mention "difference equations" in their books. The remaining books, however, are clear about the interplay between difference equations and discrete dynamical systems. It should be noted that the last two books [21, 46] made a lot of efforts to bridge the gap between the two schools. However, a larger and more sustained efforts are needed to integrate both schools and consolidate their centrality in mathematics.

As research in difference equations takes root and more and more prominent mathematicians join in, boundaries between the two schools are increasingly blurred. Intrinsic discrete models in biology and economics are being studied without going through differential equations [12, 57, 65]. It has been argued recently that difference equations are the right medium to model physical phenomena [4, 22, 72, 75]. The question of whether time is discrete has been settled affirmatively for quite some time in quantum mechanics, and recently advocated by the above-mentioned group of mathematicians.

In this article, our main objective is to introduce to specialists and non-specialists alike the notion of discrete skew-product dynamical systems. We then specialize this to the case of periodic difference equations and apply the general results to nonautonomous Beverton–Holt equations [5, 13, 23, 24, 25, 26, 38, 41]. In addition, we will survey the recent literature on periodic difference equations [10, 9, 29, 30, 32, 33, 35, 36, 40, 49, 67, 76, 77].

The final section includes some further developments and open questions.

# 2 Discrete Dynamical Systems

Let $X$ be a topological space, $T$ a topological group, and let $\pi : X \times T \to X$. Then the triple $(X, T, \pi)$ or just $\pi$ is called a dynamical system if $\pi$ is continuous and

(i)

$$\pi(x, 0) = x \quad \text{for all } x \in X, \text{ where } 0 \text{ is the identity of } T, \tag{2.1}$$

(ii)

$$\pi(\pi(x, s), t) = \pi(x, s + t). \tag{2.2}$$

If $T$ is a topological semigroup, then $\pi$ is called a semi-dynamical system.

There are two important examples that are of general interest.

a. $T = \mathbb{R}(\mathbb{R}^+)$, the space of real numbers (nonnegative real numbers). Consider the autonomous differential equation

$$x' = f(x), \quad x(0) = x_0. \tag{2.3}$$

We assume that Eq. (2.3) has a unique solution $x(t, x_0)$. Define $\pi(x, t) \doteq x(t, x_0)$. Then $\pi : X \times \mathbb{R} \to X$ defines a continuous dynamical system. If, however, the solution $x(t, x_0)$ is defined only on the set of nonnegative real numbers, then $\pi : X \times \mathbb{R}^+ \to X$ defines a continuous semi-dynamical system.

b. $T = \mathbb{Z}(\mathbb{Z}^+)$, the space of integers (nonnegative integers). Consider the difference equation

$$x(n+1) = f(x(n)), \quad x(0) = x_0, \tag{2.4}$$

where $f : X \to X$ is at least continuous.

Let $\pi(x_0, n) = x(n)$. Then $\pi$ defines either a discrete dynamical system $\pi : X \times \mathbb{Z} \to X$ or a discrete semi-dynamical system $\pi : X \times \mathbb{Z}^+ \to X$. Equation (2.4) may be generated by the map $f : X \to X$, by putting $x(n) = f^n(x_0)$, and thus $\pi(x_0, n) = f^n(x_0)$, where $f^n = f \circ f \circ \cdots \circ f$ is the $n$th composition of $f$.

Notice that a fixed point $x^*$ of the map $f$ is called an equilibrium point of Eq. (2.4). Similarly, the (positive) orbit of $x_0 \in X$, under the map $f$, $O^+(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\}$ is the same as the solution curve $\{x(n) : n \in \mathbb{Z}^+\}$ of Eq. (2.4).

One of the most popular, and still fascinating, examples is the logistic map/difference equation

$$x(n+1) = \mu x(n)(1 - x(n))$$

$$\text{or the map} \quad f(x) = \mu x(1 - x)$$

defined on the closed interval $[0, 1]$. Due to the centrality of this example in the modern theory of difference equations, I have presented it, with great detail, in my two books [20, 21]. The reader may find other interesting expositions in many of the other books mentioned above.

# 3 Skew-Product Dynamical Systems

To motivate the notion of skew-product dynamical systems, let us look at the following simple but illustrative example.

**Example 3.1.** *Consider the nonautonomous difference equation*

$$x(n+1) = (-1)^n \left(1 + \frac{1}{n+1}\right) x(n), \quad x(0) = x_0, \tag{3.1}$$

*where* $n \in \mathbb{Z}^+$.

*The solution of Eq.* (3.1) *is given by [20]*

$$x(n) = (-1)^{\frac{n(n-1)}{2}}(n+1)x_0. \tag{3.2}$$

*If we let* $\pi(x_0, n) = x(n) = (-1)^{\frac{n(n-1)}{2}}(n+1)x_0$, *then*

$$\pi(\pi(x_0, s), t) = (-1)^{\frac{t(t-1)}{2}}(-1)^{\frac{s(s-1)}{2}}(t+1)(s+1)x_0.$$

*However*

$$\begin{aligned}\pi(x_0, s+t) &= (-1)^{\frac{(s+t)(s+t-1)}{2}}(s+t+1)x_0 \\ &\neq \pi(\pi(x_0, s), t).\end{aligned}$$

*Hence* $\pi$ *is not a semi-dynamical system as the semigroup property* (2.2) *fails to hold.*

*Let us write* $F(n,x) = (-1)^n\left(1+\frac{1}{n+1}\right)x$. *For each* $t \in \mathbb{Z}^+$, *let* $F_t(n,x) = F(n+t,x) = (-1)^{n+t}\left(1+\frac{1}{n+t+1}\right)x$. *In the space of continuous functions* $C(\mathbb{Z}^+ \times X, X)$, $X = \mathbb{R}$, *the hull of* $F$ *is defined as the closure of the translates of* $F(n,.)$: $\mathcal{H}(F) = cl\{F_t(n,.) \ : \ t \in \mathbb{Z}^+\}$. *Now* $\mathcal{H}(F) = \{F_t(n,.) \ : \ t \in \mathbb{Z}^+\} \cup \{G(n,.)\}$, *where* $G(n,x) = (-1)^n x$ *is the omega limit set of* $\{F_t(n,.) \ : \ t \in \mathbb{Z}^+\}$. *Let us now switch to the friendly notation: for each* $n \in \mathbb{Z}^+$, *put* $f_n(x) = F(n,x)$, $g_n(x) = G(n,x)$. *Then* $g_n$ *is periodic :* $g_0 = g_{2n}$, $g_1 = g_{2n+1}$, *for all* $n \in \mathbb{Z}^+$, $g_0(x) = x$, $g_1(x) = -x$.

*Define* $\pi \ : \ \mathcal{H}(F) \times \mathbb{Z}^+ \to \mathcal{H}(F)$ *as*

$$\pi((x, f_i), n) = (f_{i+n-1} \circ \cdots \circ f_{i+1} \circ f_i(x), f_{i+n}).$$

*Then*

$$\begin{aligned}\pi(\pi((x,f_i),s),t) &= \pi((f_{i+s-1} \circ \cdots \circ f_{i+1} \circ f_i(x), f_{i+s}), t) \\ &= (f_{i+s+t-1} \circ \cdots \circ f_{i+s} \circ f_{i+s-1} \circ \cdots \circ f_{i+1} \circ f_i(x), f_{i+s+t}) \\ &= \pi((x, f_i), s+t).\end{aligned}$$

*And similarly for* $g_i$.

The above scheme illustrates the essence of the notion of skew-product. To explain the general situation, consider the nonautonomous difference equation

$$x(n+1) = F(n, x(n)) \tag{3.3}$$

where $F(n,.) \in C(\mathbb{Z}^+ \times X, X) = C$. The space $C$ is equipped with the topology of uniform convergence on compact subsets of $\mathbb{Z}^+ \times X$. Let $F_t(n,.) = F(t+n,.)$ and $\mathcal{A} = \{F_t(n,.) \ : \ t \in \mathbb{Z}^+\}$ be the set of translates of $F$ in $C$. Then $G(n,.) \in \omega(\mathcal{A})$, the omega limit set of $\mathcal{A}$, if for each $n \in \mathbb{Z}^+$,

$$|F_t(n,x) - G(n,x)| \to 0$$

uniformly for $x$ in compact subsets of $X$, as $t \to \infty$ along some subsequence $\{t_{g_n}\}$. The closure of $\mathcal{A}$ in $C$ is called the hull of $F(n,.)$ and is denoted by $Y = \mathcal{H}(\mathcal{A})$. Let $\sigma \ : \ Y \times \mathbb{Z} \to Y$ be defined as the shift map : $\sigma(G(n,.),1) = G(n+1,.)$, or more generally, $\sigma(G(n,.),m) = G(n+m,.)$. Then $\sigma$ is a discrete dynamical system on $Y$.

The skew product dynamical system is now defined as $\pi \ : \ X \times Y \times \mathbb{Z}^+ \to X \times Y$, such that

$$\pi((x, G(s,.)), t) = (\Phi_t(s)(x), G_t(s,.)),$$

where $\Phi_t(s) = g_{s+t-1} \circ \cdots \circ g_{s+1} \circ g_s$, $G_t(s,.) = g_{s+t}$. If $p$ is the projection map, $p(a,b) = b$, then the following commuting diagram illustrates the notion of skew product discrete semi-dynamical system.

$$\begin{array}{ccc}
X \times Y \times \mathbb{Z}^+ & \xrightarrow{\pi} & X \times Y \\
\Big\downarrow{\scriptstyle p \times id} & & \Big\downarrow{\scriptstyle p} \\
Y \times \mathbb{Z}^+ & \xrightarrow{\sigma} & Y
\end{array}$$

For each $G(n,.) \in Y$, we define the fiber $\mathcal{F}_g$ over $G$ as $\mathcal{F}_g = p^{-1}(g)$. If $g = f_i$, then we write $\mathcal{F}_g$ as $\mathcal{F}_i$.

# 4 Periodic Difference Equations

In this section we turn our attention to the case when the map $F(n,x)$ is periodic of minimal period say $p > 1$, that is $F(n+p,.) = F(n,.)$ for all $n \in \mathbb{Z}^+$. We found it rather informative and useful if we switch to writing $F(n,x)$ as $f_n(x)$ and hence Eq. (3.2) may be written in the convenient form

$$x(n+1) = f_n(x(n)), \quad n \in \mathbb{Z}^+ \tag{4.1}$$

where it is assumed that $f_{n+p} = f_n$ for all $n \in \mathbb{Z}^+$. A point $x^*$ is a fixed point of Eq. (4.1) if $f_n(x^*) = x^*$ for all $n \in \mathbb{Z}^+$. For periodic points or cycles we have to be extra careful in defining them. For there are two main types of periodic points, one in the space $X$ and another in the product space $X \times Y$. A point $(x_0, g)$ is a periodic point with period $k$ in the skew product system $\pi$ if $\pi((x_0, g), k+n) = \pi((x_0, g), n)$ for all $n \in \mathbb{Z}^+$. And by the semigroup property this is equivalent to $\pi((x_0, g), k) = (x_0, g)$.

We now give the definition of periodicity in the space $X$ that is consistent with that in the product space.

**Definition 4.1.** *Let $c_r = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$ be a set of points in $X$, with $r \geq 1$. Then $c_r$ is said to be a geometric $r$-cycle if for $i = 0, 1, \ldots, r-1$*

$$f_{(i+nr) \bmod p}(\bar{x}_i) = \bar{x}_{(i+1) \bmod r}, \quad n \in \mathbb{Z}^+. \tag{4.2}$$

The following example illustrates the "layout" in $X \times Y$ of the orbit $(\bar{x}_0, f_0)$ under the action of the skew product semi-dynamical system $\pi$.

**Example 4.2.** *Define*

$$\begin{aligned}
&f_0(x) = 1 - x, \\
&f_2(x) = f_0(x) \text{ for } x \leq 1 \text{ and } = 0 \text{ otherwise}, \\
&f_4(x) = f_0(x) \text{ for } x \leq 1 \text{ and } = 0 \text{ otherwise}, \\
&f_1(x) = x, \\
&f_3(x) = f_1(x) \text{ for } x \leq 1 \text{ and } = 0 \text{ otherwise}, \\
&f_5(x) = f_1(x) \text{ for } x \leq 1 \text{ and } = 0 \text{ otherwise}, \\
&f_n(x) = f_{n \bmod 6}, \ n \geq 6
\end{aligned}$$

*Then the system $\sigma$ on the base $Y = \{f_0, f_1, \ldots, f_5\}$ is defined by $\sigma(f_i, n) = f_{(i+n) \bmod 6}$. Notice that $c_4 = \{0, 1, 1, 0\}$ is a geometric 4-cycle, which gives rise to a 12-cycle in the skew-product system: $\{(0, f_0), (1, f_1), (1, f_2), (0, f_3), (0, f_4), (1, f_5), (1, f_0), (0, f_1), (0, f_2), (1, f_3), (1, f_4), (0, f_5)\}$. Observe that the period of the cycle in the skew-product system is the least common multiple $[4, 6] = 12$ of $r = 4$ and $p = 6$. Furthermore, the number of distinct points in each fiber is $[4, 6]/6 = 2$ each of which is of period 12 in the skew product $\pi$ (Figure 1).*

The following crucial lemma formalizes the above discussion.

**Lemma 4.3.** *[23] Let $c_r = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$ be a geometric $r$-cycle. Then the $\pi$-orbit intersects each fiber $\mathcal{F}_i$, $0 \leq i \leq p-1$, in exactly $l = s/p$, where $s = [r, p]$ is the least common multiple of $r$ and $p$, and each of these points is periodic under the skew-product dynamical system with period $s$.*

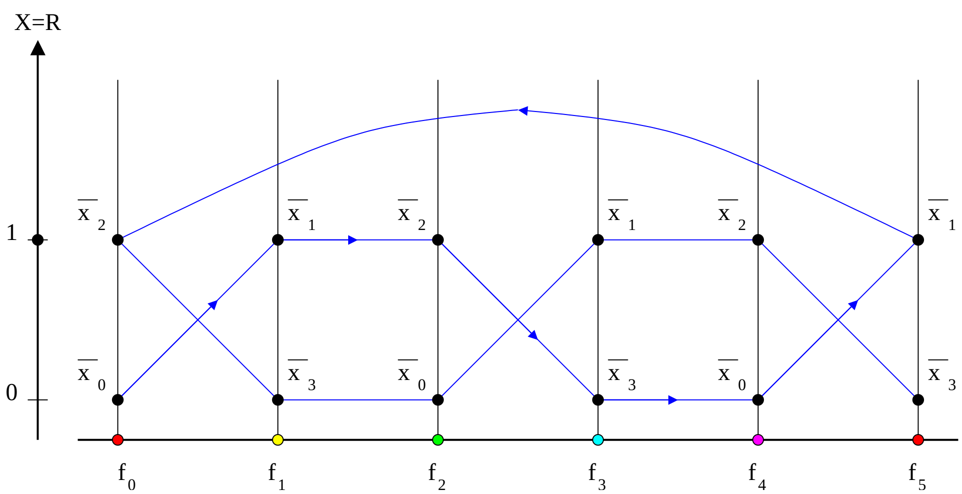


Figure 1: The period of the cycle in the skew-product is given by $s = [r, p]$, the least common multiple of $r$ and $p$, and the number of distinct points from $c_r$ in each fiber is $l = [r, p]/p$.

As a consequence of the preceding Lemma, we now state the second fundamental result in this survey, the anticipated extension of Elaydi-Yakubu Theorem to periodic difference equations.

**Theorem 4.4 (Elaydi-Sacker Theorem).** *[23] Consider the periodic difference Eq.* (4.1) *with minimal period $p$ such that each $f_i : X \to X$ is a continuous function on a connected metric space $X$. If $c_r$ is a geometric $r$-cycle which is globally asymptotically stable, then $r$ divides $p$.*

The next example shows how to utilize Theorem 4.4 to prove that a given cycle is not globally asymptotically stable.

**Example 4.5.** *Consider the two-dimensional system*

$$x(n+1) = \frac{1}{2}x(n) - \frac{\sqrt{3}}{2}y(n) + (x^2(n) + y^2(n) - 1)\cos\left(\frac{2\pi n}{9}\right)$$
$$y(n+1) = \frac{\sqrt{3}}{2}x(n) + \frac{1}{2}x(n) + (x^2(n) + y^2(n) - 1)\sin\left(\frac{2\pi n}{9}\right).$$

*This is a periodic system of period $p = 9$. The solution $(x(n), y(n)) = \left(\cos\frac{n\pi}{3}, \sin\frac{n\pi}{3}\right)$ is of period 6.*

*Since 6 does not divide 9, it follows by Theorem 4.4 that this periodic solution is not globally asymptotically stable.*

To this end, we have investigated the permissible globally asymptotically stable $r$-cycle. The next question that we will address is what are the permissible cycles, with or without any stability properties, in a given $p$-periodic difference equation?

The next result provides the definitive answer to the above question.

**Theorem 4.6.** *[25] Consider the p-periodic difference Eq.* (4.1)*. Let* $c_r = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$ *be a set of* $r$ *points in a metric space* $X$*,* $d = (r, p)$*, the greatest common divisor of* $r$ *and* $p$*, and* $m = p/d$*. Then the following statements are equivalent.*

*(a)* $c_r$ *is a geometric* $r$*-cycle of equation* (4.1)*,*

*(b)* $f_{(i+nd) \bmod p}(\bar{x}_i) = \bar{x}_{i+1}$*,* $0 \le i \le r-1$ *and* $n = 0, 1, \dots, (m-1)$*,*

*(c) the graphs of the functions* $f_i, f_{i+d}, f_{i+d}, f_{i+2d}, \dots, f_{i+(m-1)d}$ *intersect at the points* $(\bar{x}_i, \bar{x}_{(i+1) \bmod r})$*,* $(\bar{x}_{(i+d) \bmod r}, (\bar{x}_{(i+d+1) \bmod r})$*, …,* $(\bar{x}_{(i+(m-1)d) \bmod r}, \bar{x}_{(i+(m-1)d+1) \bmod r})$*, for* $0 \le i \le r-1$*.*

We remark here that if the graphs of the maps $f_i$, $0 \le i \le p-1$, are disjoint (with the exception of possibly possessing a common fixed point), then by the preceding theorem, the only possible geometric cycles are those of period $p$ or multiples of $p$.

In the next section we will discuss the implications of this fundamental observation for nonautonomous Beverton-Holt equations.

# 5 Nonautonomous Beverton–Holt Equations

Cushing and Henson [13] considered a periodically forced Beverton–Holt equation of the form

$$x(n+1) = \frac{\mu K_n x(n)}{K_n + (\mu - 1)x(n)} \tag{5.1}$$

where $\mu$ is the intrinsic growth rate and $K_n = K_{n+p}$, $n \in \mathbb{Z}^+$, is the periodic carrying capacity of a population.

Equation (5.1) is a perturbation of the original autonomous Beverton–Holt model

$$x(n+1) = \frac{\mu K x(n)}{K + (\mu - 1)x(n)}. \tag{5.2}$$

It is well known (see [20] or [15]) that if $K > 0$ and $0 < \mu < 1$, then the zero solution is globally asymptotically stable on $[0, \infty)$. Moreover, if $K > 0$ and $\mu > 1$, the positive equilibrium $x^* = K$ is globally asymptotically stable on $(0, \infty)$. The case $\mu = 1$ is trivial since then every point is a fixed point.

The following two conjectures were proposed by Cushing and Henson [13]. From now on we assume that $p \geq 2$, $K_n > 0$, $\mu > 1$.

**Conjecture 5.1.** *Equation* (5.1) *has a positive p-periodic solution which is globally asymptotically stable on* $(0, \infty)$*.*

**Conjecture 5.2.** *If* $c_p = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{p-1}\}$ *is a p-periodic solution of Eq.* (5.1)*, then*

$$av(\bar{x}_n) < av(K_n)$$

*where*

$$av(\bar{x}_n) = \frac{1}{p} \sum_{i=0}^{p-1} \bar{x}_n,$$

*and similarly for* $av(K_n)$*.*

These two conjectures were proved by Elaydi and Sacker in [23, 24]. In fact, Conjecture 5.1 was proved for a more general class of maps, called class $\mathcal{K}$. Independently, Kocic [38] solved the above two conjectures where he also considered the more general case when $K_n$ is bounded, that is $0 < \alpha < K_n < \beta < \infty$. In [41, 42], Kon proved the second conjecture for a class of systems that include the periodic Beverton–Holt equation. He considered the following class of difference equations of the form

$$x(n+1) = g\left(\frac{x(n)}{K_n}\right) x(n), \quad x(0) = x_0, \quad n \in \mathbb{Z}^+, \tag{5.3}$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies the following conditions:

(i) $g(1) = 1$,

(ii) $g(x) > 1$, for all $x \in (0,1)$, and $g(x) < 1$ for all $x \in (1,\infty)$. It is also assumed that $K_n > 0$, $K_{n+p} = K_n$ for all $n \in \mathbb{Z}^+$.

Notice that Eq. (5.3) includes Ricker's equation

$$x(n+1) = x(n)\exp\left[r\left(1 - \frac{x(n)}{K_n}\right)\right] \tag{5.4}$$

where $f_n(x) = x\exp\left[r\left(1 - \frac{x(n)}{K_n}\right)\right]$ is not monotonic (in contrast to the Beverton–Holt maps which are monotonic).

Let $K_M = \max\{K_i : 0 \le i \le p-1\}$, $K_m = \min\{K_i : 0 \le i \le p-1\}$. Suppose that the following inequality holds

$$\frac{K_M}{K_m}\exp(r-1) \le 2. \tag{5.5}$$

Zhou and Zou [77] showed that under condition (5.5), Eq. (5.4) has a globally asymptotically stable periodic solution of period $p$. In fact, the authors proved only the existence of a $p$-periodic solution which is globally attracting. However, by a Theorem of Sedaghat [66], a globally attracting fixed point in $\mathbb{R}$ is necessarily stable, and hence the above statement.

Kon [41] used this result to show that, under condition (5.5), Eq. (5.4) has a $p$-periodic solution for which Conjecture 5.2 holds. In 1990, Clark and Gross [9] discussed a discrete analogue of the nonautonomous Pearl-Verhulst logistic differential equation

$$N'(t) = r(t)N(t)[1 - N(t)/K(t)] \tag{5.6}$$

where $r(t)$ and $K(t)$ are positive, bounded periodic functions of period $T$. Their discrete model is of the form

$$x(n+1) = \frac{a_n x(n)}{1 + b_n x(n)} \tag{5.7}$$

where $a_n$ and $b_n$ are positive, bounded and periodic of integer period $p$. The authors then showed that if

$$\sum_{i=0}^{p-1} a_i > 1, \tag{5.8}$$

then Eq. (5.7) has a globally asymptotically stable $p$-periodic solution. Observe that the periodic Beverton–Holt equation may be written in the form

$$x(n+1) = \frac{\mu x(n)}{1 + \frac{(\mu-1)}{K_n} x(n)} \tag{5.9}$$

which is of the form (5.7) with $a_n = \mu$ and $b_n = (\mu - 1)/K_n$. Thus if $\mu > 1$, condition (5.8) is automatically satisfied, and the result of Clark and Gross proves Conjecture 5.2 as well. Surprisingly, most of the authors who recently tackled Cushing–Henson Conjectures, including Cushing and Henson and the authors of this paper, were not aware of this early work!

To this end, we have shown that Eq. (5.1), with $\mu > 1$, $K_n > 0$, has a globally asymptotically stable $p$-periodic solution. By the remarks after Theorem 4.6, we can now confirm that this periodic solution has a minimal period $p$. For otherwise let $r|p$, $r < p$, be the minimal period of this periodic solution. Then by Theorem 4.6

$$K_i = K_{(i+r) \bmod p} = \cdots = K_{(i+(m-1)r) \bmod p}, \quad i \in \mathbb{Z}^+$$

where $m = p/r$. This implies that Eq. (5.1) is of period $r$, a contradiction.

The situation, however, is drastically different if we assume also that $\mu = \mu_n$ is periodic with a common minimal period $p$.

Consider the equation

$$x(n+1) = \frac{\mu_n K_n x(n)}{K_n + (\mu_n - 1) x(n)}, \quad \mu_n > 1, \quad K_n > 0, \tag{5.10}$$

where both intrinsic growth rate $\mu_n$ and the carrying capacity $K_n$ are of minimal common period $p \geq 2$.

It was shown in [25] that Eq. (5.10) has a globally asymptotically stable $p$-cycle $\{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{p-1}\}$, where

$$\begin{aligned}
\bar{x}_0 &= \frac{L_{p-1}(Q_{p-1} - 1)}{E_{p-1}}, \\
L_{p-1} &= K_{p-1} \ldots K_0, \\
Q_{p-1} &= \mu_{p-1} \ldots \mu_0, \\
E_{p-1} &= K_{p-1} E_{p-2} + (\mu_{p-1} - 1)\mu_{p-2}\mu_{p-3} \ldots \mu_0 K_{p-2} K_{p-3} \ldots K_0.
\end{aligned}$$

Moreover, this $p$-periodic orbit may not be of minimal period $p$. This is in contrast to the case of Eq. (4.1) when the intrinsic growth $\mu_n$ is constant. In fact, Eq. (5.10) has a $p$-periodic orbit of minimal period $r < p$ if and only if

$$\frac{L_{p-1}(Q_{p-1}-1)}{E_{p-1}} = \frac{L_{r-1}(Q_{r-1}-1)}{E_{r-1}}.$$

The following example from [11] produces a periodic cycle whose minimal period is less than the period of the given difference equation.

**Example 5.3.** *Let $\mu_0 = 3$, $\mu_1 = 4$, $\mu_2 = 2$, $\mu_3 = 5$; $K_0 = 1$, $K_1 = \frac{6}{17}$, $K_2 = 2$, $K_3 = \frac{4}{17}$, where $\mu_n$ and $K_n$ are periodic of minimal period $p = 4$. The equation*

$$x(n+1) = \frac{\mu_n K_n}{K_n + (\mu_n - 1)x_n} x_n = f_n(x_n)$$

*has the geometric 2-cycle $c_2 = \left\{\frac{2}{5}, \frac{2}{3}\right\}$. Notice that the graphs of the maps $f_0$ and $f_2$ intersect at the point $\left(\frac{2}{5}, \frac{2}{3}\right)$, while the graphs of the maps $f_1$ and $f_3$ intersect at the points $\left(\frac{2}{3}, \frac{2}{5}\right)$.*

# 6 Further Developments

In [25], the authors investigated the extension of the second Cushing–Henson conjecture to Eq. (5.10). They obtained the following inequality:

$$av(\bar{x}_n) < \frac{\mu^*}{\mu_*} \cdot \frac{(\mu^* - 1)}{(\mu_* - 1)} av(K_n), \tag{6.1}$$

where $\mu^* = \max\{\mu_n\}$, $\mu_* = \min\{\mu_n\}$. And for the case $p = 2$, we have the following sharp result.

$$av(x_n) = av(K_n) + \sigma\left(\frac{K_0 - K_1}{2}\right) - \gamma\left(\frac{\mu_0 + \mu_1}{2}\right)(K_0 - K_1)^2, \tag{6.2}$$

where

$$\sigma = \frac{\mu_1 - \mu_0}{\mu_0\mu_1 - 1}, \quad \gamma = \frac{(\mu_0 - 1)(\mu_1 - 1)}{(\mu_0\mu_1 - 1)^2}.$$

It would be interesting to extend Eq. (6.2) to the case $p > 2$.

Recently, several authors [17, 18] considered the following autonomous difference equation with delay.

$$x(n) = f(x(n-k)), \quad k > 1. \tag{6.3}$$

If we let

$$y_j(n) = x(nk - j), \quad j = 1, 2, \ldots, k,$$

then

$$y_j(n) = f(y_j(n-1)), \quad j = 1, 2, \ldots, k$$

is a set of $k$ uncoupled first order difference equations. Hence one gains information about Eq. (6.3) by considering the associated first order difference equation

$$x(n) = f(x(n-1)). \tag{6.4}$$

In particular, [18] considered the set

$$S_k(p') = \{lp' : l|k \text{ and } (k/l, p') = 1\}.$$

They showed that if $p$ is a period of Eq. (6.4) and $p' \trianglerighteq p$ in Sharkovsky ordering of positive integers [21], i.e., $p'$ is either equal to $p$ or to the left of $p$ in the Sharkovsky ordering, then each number in the set $S_k(p')$ is a period of Eq. (6.3).

This is a nice extension of the famous Sharkovsky Theorem [21] from continuous maps on the real line to higher-order difference equations.

This leads to several questions:

1. What is the natural extension of Sharkovsky's Theorem to first order periodic difference equations?

2. What is the natural extension of Sharkovsky's Theorem to periodic difference equations with delay such as

$$x(n) = f_n(x(n-k))? \tag{6.5}$$

3. What is the natural extension of Elaydi–Sacker Theorem to periodic equations of the form (6.5)?

4. In particular, what are the extensions of the Cushing–Henson Conjectures to the equations

$$x(n) = \frac{\mu K_n x_{n-k}}{K_n + (\mu - 1)x_{n-k}} \quad \text{and} \quad x(n) = \frac{\mu K_n x_{n-k}}{K_n + (\mu_n - 1)x_{n-k}}?$$

Question 1 has been successfully addressed in [4].

# References


[1] Agarwal, R.P., *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] Al-Kahby, H., F. Dannan, and S. Elaydi, Nonstandard discretization methods for some biological models, *Applications of Nonstandard Finite Difference Schemes*, R. Mickens, ed., World Scientific, 2000.

[3] Alligood, K., T. Sauer, J. Yorke, *CHAOS: An Introduction to Dynamical Systems*, Springer–Verlag, New York, 1997.

[4] Al-Sharawi, Z., J. Angelo, and S. Elaydi, An extension of Sharkovsky Theorem to periodic difference equations, preprint.

[5] Beverton, R.J., and S.J. Holt, *The Theory of Fishing, In Sea Fisheries; Their Investigation in the United Kingdom*, M. Graham, ed., pp. 372–441, Edward Arnold, London, 1956.

[6] Birkhoff, G., Formal theory of irregular difference equations, *Acta Math.* **54** (1930), 205–246.

[7] Birkhoff, G., W.J. Trjitzinsky, Analytic theory of singular difference equations, *Acta Math.* **60** (1932), 1–89.

[8] Chihara, T.S., *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[9] Clark, M.E. and L.J. Gross, Periodic solutions to nonautonomous difference equations, *Math. Biosci.* **102** (1990), 105–119.

[10] Coleman, B.D., Nonautonomous logistic equations models of the adjustment of population to environmental changes, *Math. Biosci.* **45** (1979), 159–173.

[11] Cushing, J., *An Introduction to Structured Population Dynamics*, SIAM, Philadelphia, 1998.

[12] Cushing, J.M., R.F. Costantino, B. Dennis, R.A. Desharnais, and S.M. Henson, *Chaos in Ecology: Experimental Nonlinear Dynamics*, Academic Press, New York, 2003.

[13] Cushing, J.M. and S.M. Henson, A periodically forced Beverton-Holt equation, *J. Difference Equ. and Appl.* **8** (2002), 1119–1120.

[14] Cushing, J.M. and S.M. Henson, Global dynamics of some periodically forced, monotone difference equations. *J. Difference Equ. and Appl.* **7** (2001), 859–872.

[15] Cushing, J.M. and S.M. Henson, The effect of periodic habit fluctuations on a nonlinear insect population model, *J. Math. Biol.* **36** (1997), 201–226.

[16] Devaney, R., *A First Course in Chaotic Dynamical Systems: Theory and Experiments*, Addison-Wesley, Reading, MA, 1992.

[17] Diekman, O., and S.A. VanGils, Difference equations with delay, *Japan J. Indust. Appl. Math.* **17** (2000), 73–84.

[18] Der Heidan, U.A., and M-L. Liang, Sharkovsky orderings of higher order difference equations, *Discrete Contin. Dyn. Syst.* **11** (2004), 599–614.

[19] Edelstein-Keshet, L., *Mathematical Models in Biology*, Random House, New York, 1988.

[20] Elaydi, S., *An Introduction to Difference Equations*, Third Edition, Springer–Verlag, New York, 2004.

[21] Elaydi, S., *Discrete Chaos* Chapman & Hall/CRC, Boca Raton, 2000.

[22] Elaydi, S., Is the world evolving discretely?, *Adv. in Appl. Math.* **31(1)** (2003), 1–9.

[23] Elaydi, S. and R. Sacker, Global stability of periodic orbits of nonautonomous difference equations and population biology, *J. Differential Equations* (to appear).

[24] Elaydi, S. and R. Sacker, *Global stability of periodic orbits of nonautonomous difference equations in population biology and the Cushing-Henson conjectures,* Proceedings of the 8th International Conference on Difference Equations, Brno, 2003.

[25] Elaydi, S. and R. Sacker, Nonautonomous Beverton–Holt equations and the Cushing-Henson conjectures, *Trinity University Technical Report 91* (to appear in J. Difference Equ. Appl.).

[26] Elaydi, S. and R. Sacker, Periodic difference equations, population biology, and the Cushing–Henson conjectures, (submitted).

[27] Elaydi, S. and A.-A. Yakubu, Global stability of cycles: Lotka-Voltera competition model with stocking, *J. Difference Equ. Appl.* **8(6)** (2002), 537–549.

[28] Feigenbaum, M., Quantitative universality for a class of nonlinear transformations, *J. Statist. Phys.* **19** (1978), 25–52.

[29] Franke, J.E. and J.F. Selgrade, Attractor for periodic dynamical systems, *J. Math. Anal. Appl.* **286** (2003), 64–79.

[30] Franke, J.E. and A.-A. Yakubu, Multiple attractors via cusp bifurcation in periodically varying environments, *J. Difference Equ. Appl.* (2005).

[31] Gautschi, W., Computational aspects of three-term recurrence relations, *SIAM Rev.* **9(1)** (1967), 24–82.

[32] Grinfeld, M., P.A. Knight and H. Lamba, On the periodically perturbed logistic equation, *J. Phys. A.:Math. Gen.* **29** (1996), 8635–8040.

[33] Henson, S.M., Multiple attractors and resonance in periodically forced population, *Physica D.* **140** (2000), 33–49.

[34] Holmgren, R., *A First Course in Discrete Dynamical Systems*, Second Edition, Springer–Verlag, New York, 1996.

[35] Jillson, D., Insect populations respond to fluctuating environment, *Nature* **288** (1980), 699–700.

[36] Kapral, R. and P. Mandel, Bifurcation structure of the nonautonomous quadratic map, *Phys. Rev. A* Vol. 32, No. 2 (1985), 1076–1081.

[37] Kelley, W.G. and A.C. Peterson, *Difference Equations, An Introduction with Applications*, Second Edition, Academic Press, New York, 2000.

[38] Kocic, V.L., A note on the nonautonomous Beverton–Holt model, J. Differential Equations (to appear).

[39] Kocic, V.L. and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic, Dordrecht, 1993.

[40] Kocic, V.L., D. Stutson, and G. Arora, Global behavior of solutions of a nonautonomous delay logistic, J. Difference Equ. Appl. (to appear).

[41] Kon, R., A note on attenuant cycles of population models with periodic carrying capacity, *J. Difference Equ. Appl.* **10(8)** (2004), 791–793.

[42] Kon, R., Attenuant cycles of population models with periodic carrying capacity, *J. Difference Equ. Appl.* (to appear).

[43] Kot, M., *Elements of Mathematical Ecology*, Cambridge University Press, Cambridge, 2001.

[44] Kulenovic, M.R.S. and G. Ladas, *Dynamics of Second-Order Rational Difference Equations*, Chapman & Hall/CRC Press, Boca Raton, FL, 2002.

[45] Kulenovic, M.R.S. and G. Ladas, *Dynamics of Second-Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC Press, Boca Raton, FL, 2003.

[46] Kulenovic, M.R.S. and O. Merino, *Discrete Dynamical Systems and Difference Equations*, Chapman & Hall/CRC, Boca Raton, 2002.

[47] Kuznetsov, Y.A., *Elements of Applied Bifurcation Theory*, Springer–Verlag, New York, 1995.

[48] Lakshmikantham, V. and D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications*, Second Edition, Marcel Dekker, New York, 2002.

[49] Li, J., Periodic solutions of population models in a periodically fluctuating environment, *Math. Biosci.* **110** (1992), 17–25.

[50] Li, T.Y., and J.A. Yorke, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985–992.

[51] Liu, P. and S. Elaydi, Discrete competitive and cooperative models of Lotka-Volterra Type, *J. Comput. Anal. Appl.* **3** (2001), 53–73.

[52] Martelli, M., *Introduction to Discrete Dynamical Systems and Chaos*, John Wiley & Sons, Inc., New York, 1999.

[53] May, R., Simple mathematical models with very complicated dynamics, *j.* (1976).

[54] Mickens, R., *Difference Equations*, Van Nostrand Reinhold, New York, 1990.

[55] Mickens, R., *Applications of nonstandard finite difference schemes*, World Scientific, Singapore, 2000.

[56] Murray, J.D., *Asymptotic Analysis*, Clarendon Press, Oxford, 1974.

[57] Nishimura, K., Strategic growth, *J. Difference Equ. Appl.* **10(5)** (2004), 515–527.

[58] Perron, O., Über lineare differenzegleichungen, *Acta Math.* **(34** (1911), 109–137.

[59] Pielou, E.C., *An Introduction to Mathematical Ecology*, Wiley Interscience, New York, 1969.

[60] Pincherle, S., Delle funzioni ipergeometriche e di varie questionś ad esse attinenti, *Giorn. Mat. Battaglini* **32** (1894), 209–291.

[61] Poincaré, H., Sur les equations linéaires aux differentielles ordinaires et aux différences finies, *Amer. J. Math.* **7** (1885), 203–258.

[62] Sacker, R.J., Dedication to George Sell, *J. Difference Equ. Appl.* **9(5)** (2003), 437–440.

[63] Sacker, R.J. and G.R. Sell, Lifting properties in skew-product flows with applications to differential equations, *Mem. Amer. Math. Soc.* **11** (1977), p–p.

[64] Sandefur, J.T., *Discrete Dynamical Systems*, Clarendon Press, Oxford, 1990.

[65] Sedaghat, H., Nonlinear difference equations, in *Theory with Applications to Social Science Models*, Kluwer Academic, Dordrecht, 2003.

[66] Sedaghat, H., The impossibility of unstable globally attracting fixed points for continuous mappings of the line, *Amer. Math. Monthly* **104** (1997), 356–358.

[67] Selgrade, J.F. and H.D. Roberds, On the structure of attractors for discrete periodically forced systems with applications to population models, *Physica D.* **158** (2001), 69–82.

[68] Sell, G., *Topological Dynamics and Differential Equations*, Van Nostrand Reinhold, New York, 1971.

[69] Sharkovsky, A.N., Coexistence of cycles of a continuous transformation of a line into itself, *Ukrain. Math. Zh.* **16** (1964), 61–71 (in Russian).

[70] Smith, H., Planar competitive and cooperative difference equations, *J. Difference Equ. Appl.* **3** (1998), 335–357.

[71] Strogatz, S., Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering, *Perseus Books Group*, 2001.

[72] Tempkin, J.A. and J. Yorke, Measurements of a physical process satisfy a difference equation, *J. Difference Equ. Appl.* **8** (2002), 13–24.

[73] Van Assche, W., Asymptotics for orthogonal polynomials, *Lecture Notes in Math*, vol. 1265, Springer–Verlas, New York, 1987.

[74] Wimp, J. and D. Zeilberger, Resurrecting the asymptotics of linear recurrences, *J. Math. Anal. Appl.* **111** (1985), 162–176.

[75] Zeilberger, D., "Real" analysis is a degenerate case of discrete analysis, *Proceedings of the Sixth International Conference on Difference Equations*, Aulbach, et al, ed., CRC Press, 2004.

[76] Zhang, Q.Q. and Z. Zhou, Global attractivity of a nonautonomous discrete logistic model, *Hokkaido Math. J.* **29** (2000), 37–44.

[77] Zhou, Z. and X. Zou, Stable periodic solutions in a discrete periodic logistic equation, applications to population models, *Appl. Math. Lett.* **16** (2003), 165–171.