Periodic Difference Equations, Population Biology and the Cushing-Henson Conjectures

Saber Elaydi
Trinity University, selaydi@trinity.edu

Robert J. Sacker

Follow this and additional works at: http://digitalcommons.trinity.edu/math_faculty

Part of the Mathematics Commons

Repository Citation
Periodic difference equations, population biology and the Cushing-Henson conjectures

Saber Elaydi
Department of Mathematics
Trinity University
San Antonio, Texas 78212, USA
E-mail: selaydi@trinity.edu
http://www.trinity.edu/selaydi

Robert J. Sacker*
Department of Mathematics
University of Southern California
Los Angeles, CA 90089-2532 USA
E-mail: rsacker@math.usc.edu
http://math.usc.edu/~rsacker

Abstract

We show that for a $k$-periodic difference equation, if a periodic orbit of period $r$ is globally asymptotically stable (GAS), then $r$ must be a divisor of $k$. In particular sub-harmonic, or long periodic, oscillations cannot occur. Moreover, if $r$ divides $k$ we construct a non-autonomous dynamical system having minimum period $k$ and which has a GAS periodic orbit with minimum period $r$. Our method uses the technique of skew-product dynamical systems. Our methods are then applied to prove two conjectures of J. Cushing and S. Henson concerning a non-autonomous Beverton-Holt equation which arises in the study of the response of a population to a periodically fluctuating environmental force such as seasonal fluctuations in carrying capacity or demographic parameters like birth or death rates. We show that the periodic fluctuations in the carrying capacity always have a deleterious effect on the average population, thus answering in the affirmative the second of the conjectures. Independently Ryusuke Kon [9], [10] discovered a solution to the second conjecture and in fact proved the result for a wider class of difference equations including the Beverton-Holt equation. The work of Davydova, Diekmann and van Gils, [6] should also be noted. There they study non-linear Leslie matrix models describing the population dynamics of an age-structured semelparous species, a species whose individuals reproduce only once and die afterwards. See also the work of N.V. Davydova, [5] where the notion of families of single year class maps is introduced.

Keywords: Difference equation, global asymptotic stability, population biology

AMS 2000 Subject Classification: 39A11, 92D25

*Supported by University of Southern California, Letters Arts and Sciences Faculty Development Grant.
The author also thanks Shizuoka University, the organizers, the Japanese Ministry of Education, Culture, Sports, Science and Technology and the many societies who made this presentation possible.
1 Non-autonomous difference equations

A periodic difference equation with period \( k \)
\[
x_{n+1} = F(n, x_n), \quad F(n + k, x) = F(n, x) \quad x \in \mathbb{R}^n
\]
may be treated in the setting of skew-product dynamical systems [14], [13] by considering mappings
\[
f_n(x) = F(n, x) \quad f_i : \mathcal{F}_i \to \mathcal{F}_{i+1 \mod k}
\]
where \( \mathcal{F}_i \), the “fiber” over \( f_i \), is just a copy of \( \mathbb{R}^n \) residing over \( f_i \) and consisting of those \( x \) on which \( f_i \) acts, fig.1. Then the unit time mapping
\[
(x, f_i) \longrightarrow (f_i(x), f_{i+1 \mod k})
\]
generates a semi-dynamical system on the product space
\[
X \times Y \quad \text{where} \quad Y = \{f_0, \ldots, f_{k-1}\} \subset C, \quad X = \mathbb{R}^n \quad (1.1)
\]
where \( C \) is the space of continuous functions, fig.1. We thus study the \( k \)-periodic mapping system
\[
x_{n+1} = f_n(x_n), \quad f_{n+k} = f_n \quad (1.2)
\]
It is then not difficult to see that an *autonomous* equation \( f \) is one that leaves the fiber over \( f \) invariant, or put another way, \( f \) is a fixed point of the mapping \( f_i \longrightarrow f_{i+1 \mod k} \).

In Elaydi and Sacker [8] the concept of a “geometric \( r \)-cycle” was introduced and defined. The definition says essentially that a geometric \( r \)-cycle is the projection onto the factor \( X \) in the product space (1.1) of an \( r \)-cycle in the skew-product flow.

A geometric cycle is called globally asymptotically stable if the corresponding periodic orbit in the skew-product flow is globally asymptotically stable in the usual sense. The example in figure 1 is clearly not globally asymptotically stable. Globally asymptotically stable geometric \( r \)-cycles may be constructed using the following simple device. On \( \mathbb{R} \) define \( g(x) = 0.5x \). Then for \( r = 3 \) any \( k \geq 5 \) define
\[
\begin{align*}
  f_0 &= f_1 = \cdots = f_{k-4} \\
  f_{k-3} &= g(x) + 1 \\
  f_{k-2} &= g(x - 1) + 2 \\
  f_{k-1} &= g(x - 2)
\end{align*}
\]
The geometric 3-cycle consists of \( \{0, 1, 2\} \). However if one watches the progress of “0” in \( \mathbb{R} \) alone one will observe the (minimum period) \( k \)-cycle
\[
x_0 = 0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 1 \to 2 \to 0 = x_0,
\]
even though “0” seems at first to be fixed (imagine \( k \) very large). This can easily be generalized to
Theorem 1  Given $r \geq 1$ and $k > r + 1$ there exists a $k$ periodic mapping system having a globally asymptotically stable geometric $k$-cycle one of whose points “appears” fixed, i.e. it is fixed for $k - r$ iterations.

In general, when we have a geometric $r$-cycle with $r \leq k$, one has the following

Theorem 2 [8] Assume that $X$ is a connected metric space and each $f_i \in Y$ is a continuous map on $X$. Let $c_r = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$ be a geometric $r$-cycle of equation (1.2). If $c_r$ is globally asymptotically stable then $r|k$, i.e. $r$ divides $k$.

Thus the geometric 4-cycle in fig.1 cannot be globally asymptotically stable.

The next theorem shows how to construct such a dynamical system given any two positive integers $r$ and $k$ with $r|k$.

Theorem 3 [8] Given any two positive integers $r$ and $k$ with $r|k$ then there exists a non-autonomous dynamical system having minimum period $k$ and which has a globally asymptotically stable geometric $r$-cycle with minimum period $r$. 
2 The Beverton-Holt Equation

The Beverton-Holt equation has been studied extensively by Jim Cushing and Shandelle Henson [3, 4]. Also known as the *Beverton-Holt stock-recruitment equation* [1], it is a model for density dependent growth which exhibits *compensation* (Neave [12]) as opposed to *over-compensation* (Clark [2]), see also (Kot [11]). The equation takes the form

\[ x_{n+1} = \frac{\mu K x_n}{K + (\mu - 1)x_n}, \quad x_0 \geq 0 \quad K > 0 \]

where \( \mu \) is the per-capita growth rate and \( K \) is the carrying capacity. It is easily shown that for \( 0 < \mu < 1 \) the equilibrium (fixed point) \( x = 0 \) is globally asymptotically stable whereas for \( \mu > 1 \) the fixed point \( K \) is globally asymptotically stable.

In [3] the authors considered a periodic carrying capacity \( K_{n+k} = K_n \) caused by a periodically (seasonally) fluctuating environment

\[ x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n} \]

Defining

\[ f_i(x) = \frac{\mu K_i x}{K_i + (\mu - 1)x} \]

we have an equation of the form (1.2) with period \( k \).

Although this is not always desirable from a qualitative point of view, we will compute a “solution” in closed form of the periodic Beverton-Holt equation.

After two iterations

\[ x_2 = f_1 \circ f_0(x_0) = \frac{\mu^2 K_1 K_0 x_0}{K_1 K_0 + (\mu - 1)M_1 x_0}. \]

and inductively after \( k \) iterations

\[ x_k = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0(x_0) = \frac{\mu^k K_{k-1} K_{k-2} \cdots K_0 x_0}{K_{k-1} K_{k-2} \cdots K_0 + (\mu - 1)M_{k-1} x_0} \] (2.1)

where \( M_n \) satisfies the 2nd order linear difference equation:

\[ M_{n+1} = K_{n+1} M_n + \mu^{n+1} K_n K_{n-1} \cdots K_0, \quad M_0 = 1. \]

Thus

\[ M_{k-1} = \prod_{j=0}^{k-2} K_{j+1} + \sum_{m=0}^{k-2} \left( \prod_{i=m+1}^{k-2} K_{i+1} \right) \mu^{m+1} K_m K_{m-1} \cdots K_0. \]

Letting \( L_{k-1} = K_{k-1} K_{k-2} \cdots K_0 \), we finally obtain (defining \( H \))

\[ H(x) = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0(x_0) = \frac{\mu^k L_{k-1} x_0}{L_{k-1} + (\mu - 1)M_{k-1} x_0}. \]
But then the difference equation, $x_{n+1} = H(x_n)$ leaves the fiber (copy of $\mathbb{R}$) invariant and is thus independent of $n$, i.e. *autonomous*! (See fig. 2 for $k = 6$).

While this may or may not be easy to glean from (2.1) we will nevertheless use (2.1) later, but only in the case $k = 2$:

$$\bar{x} = (\mu + 1) \frac{K_1 K_0}{K_1 + \mu K_0}.$$  \hfill (2.2)

The mapping $x_{n+1} = H(x_n)$ thus has the unique positive fixed point

$$\bar{x} = \frac{\mu^k - 1}{\mu - 1} \frac{L_{k-1}}{M_{k-1}}$$

which is globally asymptotically stable with respect to positive initial conditions. By Theorem 2 we have further that either $\bar{x}$ is of minimal period $k$ or of minimal period $r$ where $r \mid k$.

## 3 The Ricatti equation

We next consider the more general autonomous Ricatti equation

$$x_{n+1} = f(x_n), \quad f(x) = \frac{ax + b}{cx + d}$$
where we assume the following conditions

1. $a, c, d > 0$, $b \geq 0$
2. $ad - bc \neq 0$
3. $bc > 0$ or $a > d$

\begin{align*}
1 & \implies f : \mathbb{R}^+ \to \mathbb{R}^+ \\
2 & \implies f \text{ not a constant function} \\
3 & \implies f \text{ has a positive fixed point (Bev.-Holt if } b = 0) \\
\end{align*}

Under composition, letting

$$g(x) = \frac{\alpha x + \beta}{\gamma x + \delta},$$

one easily obtains

$$g \circ f(x) = \frac{(a\alpha + c\beta)x + (b\alpha + d\beta)}{(a\gamma + c\delta)x + (b\gamma + d\delta)}$$

from which 1, 2 and 3 in (3.1) easily follow.

We now consider the periodic Ricatti equation

$$x_{n+1} = f_n(x_n) = \frac{a_n x_n + b_n}{c_n x_n + d_n}$$

where the coefficients satisfy 1, 2 and 3 in (3.1) and have period $k > 0$. Again, the function $H$ defined by

$$H(x) = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1 \circ f_0$$

has the same Ricatti form and satisfies 1, 2 and 3 in (3.1). Thus we conclude that the periodic Ricatti equation has a globally asymptotically stable geometric $r$-cycle and by Theorem 2, $r | k$.

## 4 The General Case

In the previous sections we based our analysis on the special form the difference equations had. In this section we extract the salient properties that makes it all work. Recall that $h : \mathbb{R}^+ \to \mathbb{R}^+$ is concave if

$$h(\alpha x + \beta y) \geq \alpha h(x) + \beta h(y) \quad \text{for all } x, y \in \mathbb{R}^+$$

where $\alpha, \beta \geq 0, \alpha + \beta = 1$. The following property is easily verified: If $f, g$ are concave and $f$ is increasing then $f \circ g$ is concave. Note however that by requiring our maps to take values in $\mathbb{R}^+$ and to be defined on all of $\mathbb{R}^+$, a concave function is automatically increasing.

Define the class $\mathcal{K}$ to be all functions which satisfy
(1) $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous

(2) $f$ is concave (and therefore increasing)

(3) There exist $x_1$ and $x_2$ such that $f(x_1) > x_1$ and $f(x_2) < x_2$, i.e. the graph of $f$ crosses the “diagonal”.

$\mathcal{K}$ generalizes the class “A1” of Cushing and Henson [3].

Properties of $\mathcal{K}$.

(a) $\mathcal{K}$ is closed under the operation of composition, i.e. $f, g \in \mathcal{K}$ implies $f \circ g \in \mathcal{K}$. Thus $\mathcal{K}$ is a semi-group under composition.

(b) Each $f$ has a unique globally asymptotically stable fixed point $x_f > 0$

(c) If $f, g \in \mathcal{K}$ with $x_f < x_g$ then $x_f < x_{f \circ g} < x_g$ and $x_f < x_{g \circ f} < x_g$

Thus, for the $k$-periodic difference equation

$$x_{n+1} = F(n, x_n), \quad x \in \mathbb{R}$$

(4.1)

if for all $n, f_n \in \mathcal{K}$, where $f_n(x) = F(n, x)$ then $g$ defined by

$$g(x) = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1 \circ f_0 \in \mathcal{K}$$

represents an autonomous equation

$$x_{n+1} = g(x_n)$$

having a unique globally asymptotically stable fixed point. Therefore the difference equation (4.1) has a globally asymptotically stable geometric $r$-cycle and by Theorem 2, $r | k$.

5 The Cushing and Henson Conjectures

In [4], Cushing and Henson conjectured that for the periodic $k$-Beverton-Holt equation, $k \geq 2$

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n}, \quad \mu > 1, \quad K_n > 0,$$

[Conj.1] There is a positive $k$-periodic solution $\{\bar{x}_0, \ldots, \bar{x}_{k-1}\}$ and it globally attracts all positive solutions
[Conj.2] The average over \( n \) values \( \{y_0, y_1, \ldots, y_{n-1}\} \), \( av(y_n) = \frac{1}{k} \sum_{i=0}^{k-1} y_i \) satisfies
\[
 av(\bar{x}_n) < av(K_n)
\]
In Conj.2 it is implicit that the minimal period of the cycle \( \{\bar{x}_0, \ldots, \bar{x}_{k-1}\} \) exceeds one, i.e. it is not a fixed point. The truth of Conj.2 implies that a fluctuating habitat has a deleterious effect on the population in the sense that the average population is less in a periodically oscillating habitat than it is in a constant habitat with the same average. Earlier [3] they proved both statements for \( k = 2 \). By our remarks in the previous section, Conj.1 is now completely solved: There exists a positive \( r \)-periodic globally asymptotically stable solution, with respect to \((0, \infty)\), and moreover \( r|k \).

We now answer Conj.2 in the affirmative for all \( k \geq 2 \) and in the process give a much simpler proof in the \( k = 2 \) case.

**Comment:** Without loss of generality we will now assume that \( k \) is the minimal period. Then for the periodic sequence \( \{K_0, K_1, \ldots, K_{k-1}, K_k = K_0\} \), it follows that \( K_i \neq K_{i+1} \) for at least one \( i \in \{0,1,\ldots,k-2\} \).

Everything then follows from an elementary algebraic lemma:

**Lemma 4** Assume \( \alpha, \beta, x, y \in (0, \infty), \alpha + \beta = 1 \). Then
\[
\frac{xy}{\alpha x + \beta y} - \beta x - \alpha y = -\frac{\alpha \beta (x - y)^2}{\alpha x + \beta y}
\]  
(5.1)

**Proof** Letting \( g(x, y) \) represent the left side of (5.1), we have
\[
(\alpha x + \beta y)g(x, y) = \{(1 - \alpha^2 - \beta^2)xy - \alpha \beta (x^2 + y^2)\} = -\alpha \beta (x - y)^2.
\]

We first derive a formula for a fixed point, \( x_{fog} \) of the composition of 2 Beverton-Holt functions using the formula (2.2):
\[
f(x) = \frac{\mu K x}{K + (\mu - 1)x}, \quad g(x) = \frac{\mu L x}{L + (\mu - 1)x}
\]
\[
x_{fog} = (1 + \mu) \frac{KL}{K + \mu L} = \frac{KL}{\alpha K + \beta L} = \frac{x_f x_g}{\alpha x_f + \beta x_g}
\]  
(5.2)

where \( \alpha = 1/(\mu + 1) \) and \( \beta = \mu/(\mu + 1) \). From the previous comment and the lemma it follows that
\[
x_{fog} = \frac{x_f x_g}{\alpha x_f + \beta x_g} \leq \beta x_g + \alpha x_f,
\]  
(5.3)

with strict inequality for at least one pair \( f = f_i, g = f_{i+1}, \ i \in \{0,1,\ldots,k-2\} \).
Proof of Conj.2 for k=2:

\[ f_0(x) = \frac{\mu K_0 x}{K_0 + (\mu - 1)x}, \quad f_1(x) = \frac{\mu K_1 x}{K_1 + (\mu - 1)x} \]

\[ f_1 \circ f_0(x_0) = x_0 = x_{f_1 \circ f_0} = \frac{K_1 K_0}{\alpha K_1 + \beta K_0} \]

\[ f_0 \circ f_1(x_1) = x_1 = x_{f_0 \circ f_1} = \frac{K_0 K_1}{\alpha K_0 + \beta K_1} \]

Now add and use (5.3) and \( \alpha + \beta = 1 \),

\[ x_0 + x_1 = \frac{K_1 K_0}{\alpha K_1 + \beta K_0} + \frac{K_0 K_1}{\alpha K_0 + \beta K_1} \]

\[ < \alpha K_0 + \beta K_1 + \alpha K_1 + \beta K_0 \]

\[ = K_0 + K_1. \quad \text{QED} \]

Proof for \( k > 2 \):

I. For \( k \text{ odd} \), assume, inductively that Conj.2 is true for \((k + 1)/2\).

Take \( k=3 \). Then

\[ f_2 \circ (f_1 \circ f_0)(x_0) = x_0, \quad f_0 \circ (f_2 \circ f_1)(x_1) = x_1 \]

\[ f_1 \circ (f_0 \circ f_2)(x_2) = x_2. \]

Using the period 2 result,

1. \[ f_1 \circ f_0(x_0) = x_2, \quad \text{and} \quad f_2(x_2) = x_0 \implies \]

\[ x_0 + x_2 \leq x_{f_1 \circ f_0} + x_{f_2} = \frac{K_1 K_0}{\alpha K_1 + \beta K_0} + K_2 \]

\[ \leq \alpha K_0 + \beta K_1 + K_2 \]

2. \[ f_2 \circ f_1(x_1) = x_0, \quad \text{and} \quad f_0(x_0) = x_1 \implies \]

\[ x_0 + x_1 \leq x_{f_2 \circ f_1} + x_{f_0} \leq \alpha K_1 + \beta K_2 + K_0 \]

3. \[ f_0 \circ f_2(x_2) = x_1, \quad \text{and} \quad f_1(x_1) = x_2 \implies \]

\[ x_1 + x_2 \leq x_{f_0 \circ f_2} + x_{f_1} \leq \alpha K_2 + \beta K_0 + K_1 \]

where at least one of the inequalities is strict. Adding, using (5.3) and \( \alpha + \beta = 1 \) we get

\[ 2(x_0 + x_1 + x_2) < 2(K_0 + K_1 + K_2). \]

Sketch for \( k=5 \): (fig.3) Using the result for \( k = 3 \),

\[ f_1 \circ f_0(x_0) = x_2, \quad f_3 \circ f_2(x_2) = x_4, \]

\[ f_4(x_4) = x_0 \implies \]

\[ x_0 + x_2 + x_4 \leq x_{f_1 \circ f_0} + x_{f_3 \circ f_2} + x_{f_4} \]
After one cyclic permutation:
\[ x_1 + x_3 + x_0 \leq x_{f_2 \circ f_1} + x_{f_4 \circ f_3} + x_{f_0}. \]

After 3 more cyclic permutation:
\[
\begin{align*}
  x_2 + x_4 + x_1 & \leq x_{f_3 \circ f_2} + x_{f_0 \circ f_4} + x_{f_1} \\
  x_3 + x_0 + x_2 & \leq x_{f_4 \circ f_3} + x_{f_1 \circ f_0} + x_{f_2} \\
  x_4 + x_1 + x_3 & \leq x_{f_0 \circ f_4} + x_{f_2 \circ f_1} + x_{f_3}
\end{align*}
\]

where at least one of the inequalities is strict. Adding gives the result
\[
3(x_0 + x_1 + \cdots + x_4) < 3(K_0 + K_1 + \cdots + K_4).
\]

**Sketch for k even, k=6:** (fig.4) Inductively assume the conjecture true for \(k/2\).

\[
f_1 \circ f_0(x_0) = x_2, \quad f_3 \circ f_2(x_2) = x_4, \quad f_5 \circ f_4(x_4) = x_0 \\
\implies x_0 + x_2 + x_4 \leq x_{f_1 \circ f_0} + x_{f_3 \circ f_2} + x_{f_5 \circ f_4} \quad (5.4)
\]

\[
\leq \alpha K_0 + \beta K_1 + \alpha K_2 + \beta K_3 + \alpha K_4 + \beta K_5
\]

After one (and only one) cyclic permutation: (fig.5)
Figure 4:

\[ f_2 \circ f_1(x_1) = x_3, \quad f_4 \circ f_3(x_3) = x_5, \quad f_0 \circ f_5(x_5) = x_1 \]
\[ \implies x_1 + x_3 + x_5 \leq x_{f_2 \circ f_1} + x_{f_4 \circ f_3} + x_{f_0 \circ f_5} \]
\[ \leq \alpha K_1 + \beta K_2 + \alpha K_3 + \beta K_4 + \alpha K_5 + \beta K_0 \]

where at least one of the inequalities is strict. Adding (5.4) and (5.5), we obtain the result.

**Theorem 5** For a \( k \)-periodic Beverton-Holt equation with minimal period \( k \geq 2 \)

\[ x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n}, \quad \mu > 1, \; K_n > 0 \]

there exists a unique globally asymptotically stable \( k \)-cycle, \( C = \{\xi_0, \xi_1, \ldots, \xi_{k-1}\} \) and
\[ \text{av}(\xi_n) < \text{av}(K_n) \]

Summary of proof: **Zig-zag induction**

1. Prove it directly for \( k = 2 \),
2. \( k \) odd: True for \((k+1)/2 \implies \) True for \( k \),
3. \( k \) even: True for \( k/2 \implies \) True for \( k \).

By judiciously pairing and permuting the maps and using only the formula for the fixed point of 2 maps, (5.2), it is straightforward to write down the complete proof for any \( k > 2 \).
6 Periodically varying growth parameter

In this section we consider the case in which the period \( k = 2 \) and extend our results to a periodically varying growth parameter \( \mu \).

We begin with a lemma which follows from elementary calculus:

**Lemma 6** For \( x, a > 1 \) define

\[
u(x,a) = \frac{|a - x|}{ax - 1}.
\]

Then \( u(a,a) = 0 \leq u(x,a) < 1 \).

Letting \( x_f \) and \( \mu_f \) denote respectively the stable fixed point and growth rate of a B-H function \( f \), we derive a formula for a fixed point, \( x_{fg} \) and \( x_{gf} \) of the composition of 2 B-H functions

\[
f(x) = \frac{\mu f x_f x}{x_f + (\mu_f - 1)x}, \quad g(x) = \frac{\mu g x_g x}{x_g + (\mu_g - 1)x}
\]

\[
\mu f \mu g x_{fg} x_g x
\]

\[
g \circ f = \frac{\mu f x_f x_g x}{x_f x_g + [(\mu_f - 1)x_g + (\mu_g - 1)\mu_f x_f] x}
\]

\[
= \frac{\mu f \mu g x_{gf} x}{x_{gf} + (\mu_f \mu_g - 1)x}, \text{ where}
\]

\[
x_{gf} = \frac{x_g x_f}{(r x_g + s x_f)}, \quad r = \frac{\mu_f - 1}{\mu_f \mu_g - 1}, \text{ and } s = \frac{\mu_g - 1}{\mu_f \mu_g - 1} \mu_f.
\]

(6.1)

Clearly \( r + s = 1 \). Note that from (6.1) the composition of two B-H maps is again a B-H map with \( \mu_{gf} = \mu_f \mu_g \) and \( x_{gf} \) explicitly given. Therefore the composition has a globally
asymptotically stable fixed point. In fact the B-H maps with $\mu > 1$ form a sub semi-group of the semi-group $K$ defined in [8].

Let

$$f_0(x) = \frac{\mu_0 K_0 x}{K_0 + (\mu_0 - 1)x} \quad \text{and} \quad f_1(x) = \frac{\mu_1 K_1 x}{K_1 + (\mu_1 - 1)x}$$

(6.2)

and let $x_0$ be the fixed point of $f_1 \circ f_0$. Then we have

$$x_0 = x_{f_1 \circ f_0} = \frac{K_0 K_1}{r K_1 + s K_0} = r K_0 + s K_1 - \frac{rs(K_1 - K_0)^2}{r K_1 + s K_0}$$

(6.3)

Letting $\lambda = \mu_0 \mu_1 - 1$ and substituting from (6.1), we obtain

$$\lambda x_0 = (\mu_0 - 1)K_0 + (\mu_1 - 1)\mu_0 K_1 - \frac{(\mu_0 - 1)(\mu_1 - 1)\mu_0}{(\mu_0 - 1)K_1 + (\mu_1 - 1)\mu_0 K_0}(K_1 - K_0)^2$$

A similar expression for $x_1$ is obtained by interchanging all subscripts. Adding the two and letting $\bar{x} = (x_0 + x_1)/2$ and $\bar{K} = (K_0 + K_1)/2$ we obtain

$$\bar{x} = \bar{K} + \sigma \frac{K_0 - K_1}{2} - \Delta \frac{(\mu_0 - 1)(\mu_1 - 1)}{2(\mu_0 \mu_1 - 1)}(K_0 - K_1)^2,$$

(6.4)

where

$$\Delta = \frac{\mu_0(\mu_1^2 - 1)K_0 + \mu_1(\mu_0^2 - 1)K_1}{\mu_0(\mu_1 - 1)^2 K_0^2 + (\mu_0 - 1)(\mu_1 - 1)(\mu_0 \mu_1 + 1)K_0 K_1 + \mu_1(\mu_0 - 1)^2 K_1^2} > 0$$

(6.5)

and

$$\sigma = \frac{\mu_1 - \mu_0}{\mu_0 \mu_1 - 1}$$

and from Lemma 6, $0 \leq |\sigma| < 1$.

**Remark 7** In the case $\mu_0 = \mu_1 = \mu$ (the Cushing-Henson case), the expression (6.4) reduces to

$$\bar{x} = \bar{K} - \frac{1}{2} \frac{\mu(K_0 + K_1)}{\mu K_0^2 + (\mu^2 + 1)K_0 K_1 + \mu K_1^2}(K_1 - K_0)^2$$

which gives an exact expression for the difference in the averages.

**Remark 8** For certain values of the $\mu$’s and $K$’s the state average $\bar{x}$ can exceed $\bar{K}$. For example, for $\mu_0 = 4, \mu_1 = 2, K_0 = 11, K_1 = 7$ one obtains $\bar{x} \approx 9.23$.

The calculations, even for $k = 4$, seem daunting at best.

This paper is a report on results to appear in [7].
References


