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Is the world evolving discretely?

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I. Introduction

Difference Equations can model effectively almost all physical and artificial phenomena.

Even the highly celebrated differential system of Lorenz [6], which models a fluid:

$$x' = \sigma x + \sigma y$$

$$y' = -xz + rx - y$$

$$z' = xy - bz$$

Can be studied effectively via first order nonlinear difference equations, as we will explain in the sequel. A heat rises in a fluid from the lower warm plate to the higher cool plate. If the difference in the temperature $T_l - T_u$ is small, heat is transferred by conduction and for larger difference the fluid itself moves, in convection rolls (Figure 1).

Lorenz devised a method for studying his system by considering the successive maxima of the z− coordinate of the orbit, which is the vertical direction in the Lorenz attractor (Figure 2). Lorenz's reduction method is not entirely novel and may be traced back to late 19^{th}century discoveries of Henri Poincaré. One of Poincaré's most important innovations was a simplified way of looking

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at complicated continuous orbits. Instead of studying the entire orbit, he found that much of the important information was encoded in the points in which the orbit passed through a two−dimensional plane. The order of these intersection points defines a plane map. The plane S is defined as x_3 = constant. The Poincaré map is a planar map with $G(A) = B$ (Figure 4).

If one plots the next vertical maximum z^{n+1} as a function *f* of the current z_n, we obtain a tent−like map [4] (Figure 3).

> **Main Point**: A three−dimensional differential equation can be reduced to a one−dimensional difference equation.

II. Computational Complexity

Computer Science concerns itself mainly with the computational complexity of discrete problems. However, a large portion of Physics, Economics, Biology, and Engineering is based on continuous models. Numerical Analysis is the mathematical area that concerns itself with the discretization of the continuous models in order to be amenable to computer simulations and algorithmization. Two problems arise in this process. First, discretization may be a very expensive process and, second, computer information is typically contaminated by round off errors. To summarize, information−based complexity is the study of the computational complexity of problems for which the information is partial, contaminated, and priced. I must admit that this is the type of complexity that I am least interested in. In my view, the field of numerical analysis is not only intellectually uninteresting but it is also a waste of time, energy and resources.

The point I am trying to make here is this: starting with a discrete model is a win−win situation. But, wait a minute! Since computers can represent only digital quantities and approximate real numbers with finite precision, any computer simulation of a chaotic system is doomed to degrade increasingly the father into the future one tries to predict. Worse, still, is it possible that *chaos* is nothing more than a computer artifact that results from trying to represent a stochastic world with digital numbers? The shadowing lemma is a remarkable result that addresses some of the problems mentioned above. Anosov, in 1967, and Bowen in 1975 introduced the idea for hyperbolic invertible maps. For non−hyperbolic maps, James Yorke and his collaborators [1] extended the theory in 1990.

Let $x^{n+1} = f(x^{n})$, then $\{x_n\}$ is said to be a true orbit, a sequence $\{p_n\}$ is a δ −pseudo−orbit for f if

$|p_{n+1} - f(p_n)| < \delta$, Where δ is the noise amplitude.

Shadowing: The true orbit $\{x_n\}$ δ −shadows $\{p_n\}$ if

$| x_n - p_n | < \delta$

To contain a true orbit we construct a sequence of small parallelograms $\{M_n\}$. The parallelograms must be constructed so that the image $f(M_n)$ lies across M . Moreover, two parallel sides of each M_n are designated as expanding sides, and the images of the expanding sides of M_n must intersect the two contracting sides of M_{n+1}. (This can be assured by imposing an upper bound on the sizes of the second partial derivatives of f .)

To show that there exists a true orbit $\{x_n\}$ contained in $\{M_n\}$, $x_n \in M_n$, let γ_0 be a curve lying wholly in M_0. Then $f(\gamma_0)$ contains a curve γ_1 that lies wholly in M_1 and runs from one contracting side to the other. As a matter of fact, there exist curves γ_{n+1} contained in $f(\lambda_n)$ that lie wholly in M_{n+1}. Select any point in (the final) curve γ_N and call it x_N. Then $x_{N-1} = f^{-1}(x_N)$ lies on γ_{N-1} which lies in M_{N-1}. Continuing backwards x_n is defined to be $f^{-1}(x_{n=1})$ for $0 \le n \le N$ given then a true orbit. To find the shadowing distance, we compute the distance between p_n and the furthest point of M_n and then take the maximum of these distances along the whole trajectory (Figure 5).

III. Measurement of a physical process satisfies a difference equation.

Let $\varphi : \Re^k x[0,\infty] \rightarrow \Re^k$ be a c^1−smooth map which is a semiflow; that is $\varphi(x,\ 0) = x,\ \ \varphi(\varphi(x,s),t) = \varphi(x,s+t)$. Then φ represents a physical process and is a solution of a differential equation. Let $h : \Re^k \rightarrow \Re^k$ be a c^1−smooth measurement function such as voltmeter, thermometer, or pressure gauge.

$A \subseteq \Re^k$ is a compact invariant set for φ (an attractor of φ). For $x \in A$, we examine

h at times 0, τ, 2τ, ? and collect the measurements $h_j = h(\varphi(x, j\tau))$.

<u>Theorem (Tempkin and Yorker [7])</u>. Suppose that Boxdim (A) = D, 2D < *n* for some positive integer *n*. Then for almost every measurement function *h*, there is a continuous function *f* such that for $h_j = f(h_{j-1}, h_{j-2}, \ldots, h_{j-n})$.

Note this result assumes the absence of noise in the measurements. It is not clear what would happen if noise were introduced in the system. If measurement noise is present (errors in making individual observations) and if the errors are bounded above by $\varepsilon \geq 0$, then it is still possible that one could still find a difference equation compatible with the dynamics but may be with an error $0(\varepsilon)$. However, in the presence of dynamical noise, one must ask to what extent could a difference equation for the noiseless system be useful in prediction for the noisy one. Another problem might arise if the variable of interest may be difficult to measure directly. The question here: might the difference equation for different variables be related.

I have another point of view, which confirms Tempkin and Yorke's theorem. The continuous evolution of objects that we see is just an illusion. Our brains are actually digital and what we see is actually a fast discrete measurement. This illusion of the continuously evolving world occurs if the measurements are taking at certain threshold speed. This threshold speed has been attained in the films and the TV programs we see nowadays. Using logical arguments, Inagaki [5] has shown that it is the discreteness of our brains that makes our brain works the way they are.

III. Differential equations vs. difference equations

Poincaré−Bendixon Theorem: Let *f* be a smooth vector field of the plane, for which the equilibria $v' = f(v)$ are isolated. If the forward orbit $\varphi(t,\ v_0),\quad t \geq 0$ is bounded, then either:

(i) $\omega(v_0)$ is an equilibrium, or

(ii) $\omega(v_0)$ is a periodic orbit, or

(iii) For each $u \in \omega(v_0)$, $\alpha(u)$ and $\omega(u)$ are equilibria.

If the assumption that the equilibria are isolated is removed, then we have to include the possibility that either $\omega(v_0)$ or $\omega(u)$ is a connected set of equilibria. The existence of a globally attracting cycle is possible. Take for example, the system (in polar coordinate)

$$r' = r(a - r)$$
$$\sigma' = b$$

where $a, b > 0$

The circle $r = a$ is globally attracting (every nonzero orbit converges to the limit cycle $r = a$). The main advantage here is that the Jordan Curve Theorem applies. The Jordan curve theorem says that a simple closed curve divides the plane into two parts: a bounded region (the inside) and unbounded region (the outside). In order for a path to get from a point inside the curve to a point outside the curve, it must cross the curve. Note that a periodic orbit or a cycle is a simple closed curve

Difference Equations:

There are plethora of differences between the qualitative behavior of difference equations and differential equations. In fact, it is a good project to write a book on this subject; but there are few important differences that we are reporting here.

(A) The presence of eventually equilibria or cycles in difference equations. Here in finite time an orbit may reach an equilibrium point or a cycle.

Example [4] (the tent map) .

$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \le 1 \end{cases}$$

Notice that :

$$\frac{1}{6} \to \frac{1}{3} \to \frac{2}{3} \quad \text{(Eventually fixed point).}$$

$$\frac{1}{20} \to \frac{1}{10} \to \frac{1}{5} \to \frac{2}{5} \to \frac{4}{5} \to \frac{2}{5} \quad \text{(Eventually 2−cycle.).}$$

This phenomenon does not occur in differential equations.

Remark: A swinging pendulum would go to rest in a finite time, as it is possible in a difference equation model. On the other hand, a differential equation model does not allow this phenomenon to happen; yet another verification for the inadequacy of continuous models.

(B) The impossibility of globally attracting cycles in discrete models with a connected phase space X. Why?

The orbit of a k−cycle $\{x_0, \; x_1, \; .... \; x_{k-1}\}$ is finite. If the cycle is attracting, then for each point x_i, the basin of attraction $W^s(x_i)$ is open and invariant. If the cycle is globally attracting, then the phase space X is the finite union of disjoint open sets, which violates the connectedness of X. In contrast, the orbit of a cycle in a differential equation is a closed curve with infinitely many points and infinitely many corresponding basins of attraction. In this case the connected phase space is the infinite union of disjoint open sets, which is topologically valid.

Remark: This may be another confirmation of the discreteness of our world arrow of time. Economists have known for a long time that there are no globally attracting economic cycles. The question still remains about the structure of the complement of the basin of attraction and how big it is.

The following model of two competing species confirms one more time that there are no globally attracting cycles in nature.

Example: Consider Elaydi−Yakubu Model [3] of a Lotka−Volterra competition model of two species:

$$x_{n+1} = x_n \left[\exp\left(p_1 - q_1\left(x_n + y_n\right)\right) + \alpha\right]$$

$$y_{n+1} = y_n \left[\exp\left(p_2 - q_2\left(x_n + y_n\right)\right)\right]$$

$\alpha \in (0, 1)$ is the planting coefficient.

We have conjectured that this model has a globally attracting (positive) 2−cycle. After a month of simulations, we became skeptical about the conjecture, which led to the simple observations mentioned above. An experiment is waiting to verify our conclusions. In Figure 5 we plotted the phase space of the second iterates of our map. The points in the 2−cycle are fixed points under the second iterate of the map. Here we put

$q_1 = q_2 = 1,\ p_1 = 1.5, \alpha = 0.5$, and we let p_2 vary. The gray stripes constitute the basin of attraction of one point in the 2−cycle and the white stripes are the basin of attraction of the other point in the cycle. The black region is the compliment of the basin of attraction of the 2−cycle (Figure 6).

IV Chaoplexity: a new paradigm.

We may look at DNA sequences as a symbolic sequence of symbols: A↔0, C↔1, T↔2, G↔3. Hence we have $\sum_4$, the space of four symbols that now represents the DNA sequences. Two sequences $x = x_0x_1x_2$? and $y = y_0y_1y_2$? are at distant

$$d(x, y) = \sum_{i=0}^{\infty} |x_i - y_i| / 4\wedge i$$.

Equipped with this metric, $\sum_4$ becomes a metric space. A mutation map $\sigma : \sum_4 \to \sum_4$ is defined using the Fibonacci sequence rule,

x = 2103120231?→ 32103120231?→332103120231?→2332103120231?

y = 2133120231?→32133120231?→232133120231?→1232133120231?

Notice that the sequences x and y differ only in one component. But repeated application of σ leads to increasing differences between these two sequences. This phenomenon is one of the hallmarks of chaotic systems and is called the butterfly effect or sensitive dependence on initial conditions.

Many theories have been developed addressing the question of the extinction of species. I speculate that the extinct species were not chaotic enough to survive major calamities. To demonstrate my point, let us look at a recent study by Jim Cushing and his collaborators, the beetles. In their study of the flour beetles, they developed a 3−dimensional model of difference equations to represent Larvae (L), Pupae (P) and Adults (A). They have shown both experimentally and mathematically that the flour beetles exhibit chaos through double bifurcation. So you try to exterminate them but they keep coming back in greater quantities. Plant pests have shown the same behavior. However dinosaurs were too stable to survive and perhaps the spotted owl is also non chaotic and will not survive either.

Darwin’s theory, the survival for the fittest may apply perhaps to individuals in a species. But for a species as a whole to survive it has to possess chaoplexity that is it must be both complex and chaotic. Notice that complexity is not sufficient for survival but it is necessary. Moreover, the presence of chaos is a sufficient condition for survivability but perhaps it is not a necessary condition. This definition of chaoplexity is different from that of John Doyle [2].

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