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Reduction of Jump Systems

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A jump system is a set of integer lattice points satisfying an exchange axiom. We discuss an operation on lattice points, called reduction, that preserves the jump system two-step axiom. We use reduction to prove a weakened version of a matroid conjecture by Rota[3], as well as demonstrate new operations on matroids and delta-matroids.

Key Words: jump system, delta-matroid, matroid

1. INTRODUCTION

Matroids have long been an important structure in pure combinatorics. We recall that a matroid $(E, \mathcal{M})$ consists of a finite set of edges, E, together with $\mathcal{M}$, a collection of subsets of E, satisfying an exchange axiom.

Axiom 1 (Matroid). For all $A, B \in \mathcal{M}$, and for all $e \in A \setminus B$, there is some $f \in B \setminus A$ with $A \Delta \{e, f\} \in \mathcal{M}$. Here Δ denotes symmetric difference. The elements of $\mathcal{M}$ are the *bases* of the matroid.

Delta-matroids are a generalization of matroids introduced in 1987 by Bouchet [1]. A delta-matroid $(E, \mathcal{D})$ consists of a finite set of edges, E, together with $\mathcal{D}$, a collection of subsets of E, satisfying an exchange axiom.

Axiom 2 (Delta-Matroid). For all $A, B \in \mathcal{D}$, and for all $e \in A \Delta B$, there is some $f \in B \Delta A$ with $A \Delta \{e, f\} \in \mathcal{D}$. The elements of $\mathcal{D}$ are the *feasible sets* of the delta-matroid.

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It is easy to see that a matroid is precisely a delta-matroid with equicardinal feasible sets. Delta-matroids satisfy a variety of nice properties, including a greedy algorithm and polyhedral description.

Jump systems were introduced in 1995 by Bouchet and Cunningham [2] as a generalization of delta-matroids. Let E be a finite set. We now fix $\mathbf{Z}^E$ as our ground set. Fix the l^1 metric, where for $x, y \in \mathbf{Z}^E$, $d(x,y) = \sum_{i \in E} |x_i - y_i|$. Let $x \xrightarrow{y} z$ denote that $d(x,z) = 1$ and $d(x,y) > d(z,y)$. We say that z is a *step* from x toward y. A jump system J is a subset of $\mathbf{Z}^E$ satisfying a two-step axiom.

Axiom 3 (Jump System). For all $x, y \in J$ and for all $z_1 \in \mathbf{Z}^E$ with $x \xrightarrow{y} z_1$, then either $z_1 \in J$, or there is some $z_2 \in J$ with $x \xrightarrow{y} z_1 \xrightarrow{y} z_2$.

There is a simple bijection between delta-matroids $(E, \mathcal{D})$ and jump systems J on $\{0,1\}^E \subseteq \mathbf{Z}^E$. The elements of E correspond to the coordinates of $\mathbf{Z}^E$, and feasible sets correspond to characteristic vectors.

We also recall the following theorem, which gives (under a special condition) a test for membership in a jump system.

THEOREM 1.1 (Lovász [4]). *Let J be a jump system on $\mathbf{Z}^E$. Suppose that for all $x \in J$, $\|x\|_1 = \alpha$ for some fixed α. Let $v \in \mathbf{Z}^E$ with $\|v\|_1 = \alpha$. Then $v \in J$ if and only if for all $A \subseteq E$, there is some $x \in J$ with $\sum_{i \in A} x_i \geq \sum_{i \in A} v_i$.*

If the jump system is on $\{0,1\}^E$ (and hence represents a delta-matroid), then the condition of Theorem 1.1 is equivalent to the corresponding delta-matroid actually being a matroid.

We present several new operations on jump systems. In Section 2, we define reduction and present our central result, that reduction preserves Axiom 1. In Section 3, we use reduction to prove a weakened version of a matroid conjecture by Rota[3]. Finally, in Section 4, we define several other new operations on jump systems, and show that they preserve Axiom 3.

2. REDUCTION

A reduction of an integer lattice is a lower-dimensional integer lattice, related in a natural way to the original lattice.

DEFINITION 2.1. Fix $\mathbf{Z}^E$. Let $F \subseteq E$. Set $E' = \big(E \setminus F\big) \cup \{f\}$, where f is a new element ($f \notin E$). Let R denote the *reduction* map from $\mathbf{Z}^E$ to

$\mathbf{Z}^{E'}$ defined by

$$R(x)_i = \left\{ \begin{matrix} x_i & i \in E \setminus F \\ \sum\limits_{j \in F} x_j & i = f \end{matrix} \right\}.$$

Let J be a jump system on $\mathbf{Z}^E$. We have $R(J) = \{R(x) \mid x \in J\} \subseteq \mathbf{Z}^{E'}$. We say that $R(J)$ is a *reduction* of J formed by *reducing* F.

Reduction satisfies a number of nice properties. It is quite easy to see that $R(v) + R(w) = R(v + w)$, and that $\|R(v)\| = \|v\|$. Further, the composition of two reductions is a reduction. Our central result is that a reduction of a jump system is, in turn, a jump system.

THEOREM 2.1. *$R(J)$ is a jump system.*

Proof. Suppose $E = \{1, 2, \dots, n\}$. It is sufficient to prove the result for $F = \{1, 2\}$, since (as noted above) the composition of two reductions is a reduction, so reductions of $|F| > 2$ can be achieved by iterating reductions of size 2. Let the unit vectors of $\mathbf{Z}^E$ be $\{e_1, e_2, \dots, e_n\}$. Let the unit vectors of $\mathbf{Z}^{E'}$ be $\{f_0, f_3, f_4, \dots, f_n\}$, with $R(e_1) = R(e_2) = f_0, R(e_3) = f_3, \dots, R(e_n) = f_n$. Let $x, y \in R(J)$, and let $x \xrightarrow{y} x + a$, for any $a \in \mathbf{Z}^{E'}$. If $x + a \in R(J)$, then Axiom 3 is satisfied (proving the theorem), so henceforth we make the assumption that $x + a \notin R(J)$. We would like to produce some $z \in R(J)$ with $x + a \xrightarrow{y} z$. Let $u, v \in J$ with $R(u) = x, R(v) = y$. Let $b \in \mathbf{Z}^E$ with $R(b) = a$, and with $u \xrightarrow{v} u + b$ (note that if $x_0 < y_0$, then either $u_1 < v_1$ or $u_2 < v_2$ so some such b always exists). If $u + b \in J$, then $R(u + b) = x + a \in R(J)$, contradicting our assumption. Therefore, we can apply Axiom 3 to get $u \xrightarrow{v} u + b \xrightarrow{v} u + b + c$, with $u + b + c \in J$. Now $R(u + b + c) = x + a + R(c) \in R(J)$ and $x + a + R(c)$ is a step from $x + a$, since $\|R(c)\| = 1$. If it is in the direction of y, then Axiom 3 holds and the theorem follows. If $c = \pm e_i (3 \leq i \leq n)$, then $R(c)$ is in the direction of y and the theorem follows. Otherwise, without loss of generality, we assume that $c = e_1$ and $(u + b)_1 < v_1$. There are three cases to consider:

Case A. $(u + b)_1 + (u + b)_2 < v_1 + v_2$ (i.e. $(x + a)_0 < y_0$). This is the easy case, as we have $x + a \xrightarrow{y} x + a + R(c)$.

Case B. $(u + b)_1 + (u + b)_2 > v_1 + v_2$ (i.e. $(x + a)_0 > y_0$). We have $(u + b)_1 < v_1$ and $(u + b)_2 > v_2$. This case proceeds by a sequence of steps. In each step, either the process terminates and Axiom 3 holds, or a $w_i \in J$ is produced, and the process continues. However, each step gets closer to v (i.e. $||w_1 - v|| > ||w_2 - v|| > \dots$), so the process must terminate.

Step B-1. Set $w_1 = u + b + e_1 \in J$. We have $w_1 \xrightarrow{v} w_1 - e_2$. If $w_1 - e_2 \in J$, then $R(w_1 - e_2) = x + a \in R(J)$, contradicting our assumption. We can therefore apply Axiom 3 to get $w_1 \xrightarrow{v} w_1 - e_2 \xrightarrow{v} w_1 - e_2 + d_1 \in J$. If $d_1 = \pm e_i (3 \leq i \leq n)$, then $x + a \xrightarrow{y} x + a + R(d_1)$ and Axiom 3 holds. Suppose that either $d_1 = e_1$ or $d_1 = -e_2$. If $d_1 = -e_2$, then Axiom 3 holds, as $(x + a)_0 > y_0$ and $x + a \xrightarrow{y} x + a + R(d_1)$. Hence, either Axiom 3 holds, or we must have $d_1 = e_1$ and we continue to step 2.

Step B-2. Set $w_2 = u + b + e_1 - e_2 + e_1 \in J$. We have $R(w_2) = x + a + e_0$ (note that $w_2 = w_1 - e_2 + e_1$) and $w_2 \xrightarrow{v} w_2 - e_2$. If $w_2 - e_2 \in J$, then $R(w_2 - e_2) = x + a \in R(J)$, contradicting our assumption. As before, we apply Axiom 3 to get $w_2 \xrightarrow{v} w_2 - e_2 \xrightarrow{v} w_2 - e_2 + d_2 \in J$. If $d_2 = \pm e_i (3 \leq i \leq n)$, then Axiom 3 must hold. If $d_2 = -e_2$, then again Axiom 3 must hold. Hence either Axiom 3 holds, or we must have $d_2 = e_1$, and the process continues.

Case C. $(u + b)_1 + (u + b)_2 = v_1 + v_2$ (i.e. $(x + a)_0 = (y)_0$). We have $(u + b)_1 < v_1$ and $(u + b)_2 > v_2$. This case proceeds by a sequence of steps. In each step, either Axiom 3 holds, or a $w_i \in J$ is produced, and the process continues. However, each step gets closer to v (i.e. $||w_1 - v|| > ||w_2 - v|| > \dots$), so it must terminate.

Step C-1. Set $w_1 = u + b + e_1 \in J$. We have $w_1 \xrightarrow{v} w_1 - e_2$. If $w_1 - e_2 \in J$, then $R(w_1 - e_2) = x + a \in R(J)$, which contradicts our assumption. We can therefore apply Axiom 3 to get $w_1 \xrightarrow{v} w_1 - e_2 \xrightarrow{v} w_1 - e_2 + d_1 \in J$. If $d_1 = \pm e_i (3 \leq i \leq n)$, then $x + a \xrightarrow{y} x + a + R(d_1)$ and Axiom 3 holds. Otherwise, we must have $d_1 = e_1$ or $d_1 = -e_2$. Set $d_1^\star$ so that $d_1 + d_1^\star = e_1 - e_2$ (note that $R(d_1 + d_1^\star) = R(e_1 - e_2) = 0$). We have $u + b + e_1 \xrightarrow{v} u + b + e_1 - e_2 \xrightarrow{v} u + b + e_1 - e_2 + d_1$, where $u + b + e_1 \in J$ and $u + b + e_1 - e_2 + d_1 \in J$.

Step C-2. Set $w_2 = u + b + e_1 - e_2 + d_1 \in J$. If $d_1 = e_1$, then we have $(u + b + e_1 - e_2)_1 < (v)_1$, $(u + b + e_1 - e_2)_2 > (v)_2$, and hence $w_2 \xrightarrow{v} w_2 + d_1^\star$. If $d_1 = -e_2$, then a similar argument gives $w_2 \xrightarrow{v} w_2 + d_1^\star$. If $w_2 + d_1^\star \in J$, then $R(w_2 + d_1^\star) = x + a \in R(J)$, which contradicts our assumption. We therefore apply Axiom 3 to get $w_2 \xrightarrow{v} w_2 + d_1^\star \xrightarrow{v} w_2 + d_1^\star + d_2$. If $d_2 = \pm e_i (3 \leq i \leq n)$, then $x + a \xrightarrow{y} x + a + R(d_2)$, so Axiom 3 holds. Otherwise, we must have $d_2 = e_1$ or $d_2 = -e_2$, and the process continues.

∎

3. MATROID CONSEQUENCES

Let $B_1, B_2, \ldots, B_n$ be pairwise nonintersecting bases of a rank n matroid $(E, \mathcal{M})$. Rota has conjectured in [3] that there always exists an $n \times n$ matrix A, whose jth column consists of the elements of B_j, ordered in such a way that the rows of A are bases as well.

We confirm a weaker version of this conjecture, namely that for any i with $1 \le i \le n$, there always exists an $n \times n$ matrix A whose jth column consists of the elements of B_j, and that the first i rows are a disjoint union of i bases.

First, we recall some notions about matroid union. If $(E, \mathcal{M})$ is a matroid, then the union $\mathcal{M} \bigvee \mathcal{M}$ is a matroid on E, each of whose bases is a disjoint union of two bases of $\mathcal{M}$. Similarly, the union $\overset{i}{\bigvee} \mathcal{M}$ is a matroid on E each of whose bases is a disjoint union of i bases of $\mathcal{M}$ (provided one such disjoint union exists, which must be true in the context of Rota's conjecture). For more information about matroid unions, see [6] or [5].

Theorem 3.1. *Let $B_1, B_2, \ldots, B_n$ be any bases of a rank n matroid $(E, \mathcal{M})$. If $1 \le i \le n$, then there is a basis B of $\overset{i}{\bigvee} \mathcal{M}$ with $|B \cap B_j| = i$ for $1 \le j \le n$.*

Proof. Without loss of generality, we assume $|E| = n^2$. Let J be the jump system on $\mathbf{Z}^{n^2}$ corresponding to the matroid $\overset{i}{\bigvee} \mathcal{M}$. For $1 \le j \le n$, let $F_j = \{$coordinates corresponding to the elements of $B_j\}$. We now reduce each F_j to produce a jump system on $\mathbf{Z}^n$. $R(J)$ is a jump system by Theorem 2.1. If $i \cdot \vec{1} = \underbrace{(i, i, \ldots, i)}_{n} \in R(J)$, then the theorem follows.

By Theorem 1.1, $i \cdot \vec{1} \in R(J)$ if for all $A \subseteq \{1, 2, \ldots, n\}$, there is some $x \in R(J)$ with $\sum_{j \in A} x_j \ge i|A|$. Fix $A \subseteq \{1, 2, \ldots, n\}$. We define $C \subseteq \{1, 2, \ldots, n\}$ as follows:

1. If $|A| < i$, set $C = A \cup$ any other $(i - |A|)$ elements of $\{1, 2, \ldots, n\}$.
2. If $|A| \ge i$, let $C =$ any i elements of A.

By construction, $|C| = i$. Let $y \in \mathbf{Z}^E$ be the incidence vector of $\bigcup_{j \in C} B_j$. Because y corresponds to i disjoint bases, we must have $y \in J$. Set $x = R(y) \in R(J)$. Observe that for $i \notin C$, $x_i = 0$. There are two cases to consider:

1. If $|A| < i$, then $\sum_{j \in A} x_j = |A|n \ge i|A|$.

2. If $|A| \geq i$, then $\sum_{j\in A} x_j = \sum_{j\in C} x_j = in \geq i|A|$.

Hence, in either case, we have constructed $x \in R(J)$ with $\sum_{j\in A} x_j \geq i|A|$, as desired. ∎

4. OTHER JUMP SYSTEM OPERATIONS

There are a variety of operations on jump systems that preserve the jump system property. For a survey, see [2]. We recall three of them. If J is a jump system on $\mathbf{Z}^E$, and $v \in \mathbf{Z}^E$, then the *translation* $J+v = \{x+v|x \in J\}$ is a jump system.

If J is a jump system on $\mathbf{Z}^E$, and $F \subseteq E$, then the *reflection* of J in the coordinates indexed by F is a jump system. That is, the set of x', so that $x'_i = \begin{Bmatrix} x_i & i \in E \setminus F \\ -x_i & i \in F \end{Bmatrix}$, is a jump system.

Let $a_i \in \mathbf{Z} \cup \{-\infty\}$. Let $b_i \in \mathbf{Z} \cup \{\infty\}$. $\prod_{i\in E} [a_i, b_i]$ is called a *box.* The intersection of a box with a jump system J is, in turn, a jump system.

Proposition 4.1. *Let $(E, \mathcal{D})$ be a delta-matroid, $F \subseteq E$, and $0 \leq \alpha \leq |F|$. Then there is a delta-matroid $(E', \mathcal{D}')$, where each $D' \in \mathcal{D}'$ has a corresponding $D \in \mathcal{D}$, satisfying: $f \notin D' \iff |D \cap F| = \alpha$ and $f \in D' \iff |D \cap F| = \alpha + 1$. Furthermore, if $(E, \mathcal{D})$ is a matroid, then $(E', \mathcal{D}')$ is a matroid.*

Proof. We reduce F, translate by $-\alpha$ times the unit vector corresponding to f, and intersect with the box that restricts f to $[0, 1]$ and leaves all other coordinates alone. A delta-matroid $(E, \mathcal{D})$ is a matroid precisely when $|D|$ is constant for all $D \in \mathcal{D}$. Suppose that this is the case. If a delta-basis D has $\alpha \leq |D \cap F| \leq \alpha + 1$, then $|D'| = |D| - \alpha$. Otherwise, D has no corresponding D'. Consequently, all delta-bases in the image contain $r(D) - \alpha$ elements, and the image is a matroid. ∎

Observe that the rank $r(\mathcal{M}')$ of the matroid $(E', \mathcal{M}')$ either satisfies $r(\mathcal{M}') = r(\mathcal{M}) - \alpha$, or else $r(\mathcal{M}') = 0$. The latter happens when no bases B satisfy $\alpha \leq |B \cap F| \leq \alpha + 1$.

We now introduce an operation that does not respect the matroid property (equicardinality of feasible sets). A *constriction* is a map from $\mathbf{Z}^E$ to a box.

DEFINITION 4.1. Fix $\mathbf{Z}^E$. Let $\prod_{i\in E}[a_i,b_i]$ be a fixed box. Let C denote the *constriction* map from $\mathbf{Z}^E$ to $\mathbf{Z}^E$ (in fact, into the box) defined by

$$C(x)_i = \begin{Bmatrix} x_i & a_i \le x_i \le b_i \\ a_i & x_i < a_i \\ b_i & b_i < x_i \end{Bmatrix}.$$

We have $C(J) = \{C(x)|x \in J\} \subseteq \mathbf{Z}^E$. We say that $C(J)$ is a *constriction* of J into the box.

Constriction satisfies various nice properties, such as $C(v+w) = C(v) + C(w)$, and that the composition of two constrictions is a constriction. We now show that constriction preserves Axiom 3.

PROPOSITION 4.2. *Let J be a jump system on $\mathbf{Z}^E$. Let $\prod_{i\in E}[a_i,b_i]$ be a box. Then $C(J)$ is a jump system.*

Proof. Suppose that $E = \{1,2,\dots,|E|\}$. Let $M = \max_{x\in J} x_1$. Consider the box $[-\infty, M-1] \times [-\infty,\infty] \times \cdots \times [-\infty,\infty]$. It suffices to prove that $C(J)$ is a jump system for this specific box, since the composition of two constrictions is a constriction.

Let $x', y', z_1' \in C(J)$, with $x' \xrightarrow{y'} z_1'$. If $z_1' \in C(J)$, then Axiom 3 holds, and the proposition follows. Henceforth, we assume that that $z_1' \notin C(J)$. Let $x, y \in J$ be such that $C(x) = x', C(y) = y'$. Let $z_1 \in \mathbf{Z}^E$ be such that $C(z_1) = z_1'$ and $x \xrightarrow{y} z_1$. Now, we must have $z_1 \notin J$, since $z_1' \notin C(J)$. We can therefore apply Axiom 3 to get $x \xrightarrow{y} z_1 \xrightarrow{y} z_2$ with $z_2 \in J$, and hence $C(z_2) \in C(J)$. Because $z_1 \xrightarrow{y} z_2$, we must have $z_2 = z_1 + \beta$ for some $\beta = \pm e_i, 1 \le i \le |E|$. If $\beta = \pm e_i$ for $2 \le i \le |E|$, then $C(z_2) = z_1' + \beta$, $z_1' \xrightarrow{y'} C(z_2)$, and Axiom 3 is satisfied. If $\beta = \pm e_1$, then either $C(z_2) = C(z_1)$ or $C(z_2) = z_1' + \beta$. The former is impossible since we assumed that $C(z_2) = z_1' \notin C(J)$. Therefore, we must have $z_1' \xrightarrow{y'} C(z_2)$, and the proof is complete. ▮

Let $(E, \mathcal{D})$ be a delta-matroid. Let $x, y \in E$. If we reduce $\{x, y\}$ and constrict that coordinate to $[0,1]$, then we will have a delta-matroid $((E\backslash x\backslash y)\cup z, \mathcal{D}')$, where $z \in D'$ if and only if $(x \text{ OR } y) \in D$. If we translate by the negative of the unit vector corresponding to z before constricting, we can replace OR by AND. Reflections allow us to construct any possible truth table for these two elements. Further combinations allow us to construct truth tables for any number of elements.

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