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Generating functions and Wilf equivalence for generalized interval embeddings

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Abstract

In 1999 in [*J. Difference Equ. Appl.* 5, 355–377], Noonan and Zeilberger extended the Goulden-Jackson Cluster Method to find generating functions of word factors. Then in 2009 in [*Electron. J. Combin.* 16(2), RZZ], Kitaev, Liese, Remmel and Sagan found generating functions for word embeddings and proved several results on Wilf-equivalence in that setting. In this article, the authors focus on *generalized interval embeddings*, which encapsulate both factors and embeddings, as well as the "space between" these two ideas. The authors present some results in the most general case of interval embeddings. Two special cases of interval embeddings are also discussed, as well as their relationship to results in previous works in the area of pattern avoidance in words.

1 Introduction

Let a word w be comprised of letters $w_1, w_2, w_3, \ldots \in \mathcal{P}$, where $\mathcal{P}$ is any set, called our *alphabet*. Define the Kleene closure of $\mathcal{P}$ as

$$\mathcal{P}^* = \{w = w_1 w_2 \cdots w_n \mid n \geq 0 \text{ and } w_i \in \mathcal{P} \text{ for all } i\}.$$

In this paper we set $\mathcal{P} = \mathbb{N} = \{1, 2, 3, \ldots\}$, so that any word is simply a string of positive integers. Let $|w|$ denote the number of letters in w and $\Sigma w = \sum_{i=1}^{|w|} w_i$, and we define the *weight of* w to be $\mathrm{wt}(w) = t^{|w|} x^{\Sigma w}$. In the case where $n = 0$, we define $w = \epsilon$ to be the empty word, and we have that $\mathrm{wt}(\epsilon) = 1$. Noonan and Zeilberger [6] define u to be a factor of w if there is a string, v, of $|u|$ consecutive letters in w such that for all $1 \leq i \leq |u|$, $u_i = v_i$, where u_i and v_i are the i-th letters of u and v, respectively. For example, if $u = 1443$ and $w = 841443117$, then w has u as a factor since $w_3 w_4 w_5 w_6 = u$. Later, Kitaev, Liese, Remmel, and Sagan [2] define that w *embeds* u if there is a string, v, of $|u|$ consecutive letters in w such that for all $1 \leq i \leq |u|$, $u_i \leq v_i$. For example, if $u = 154$ and $w = 16563$, then w embeds u in two places, and 165 and 656 are said to be *embeddings* of u into w. Further generalizations of this notion of embedding can be found in Langley, Liese, and Remmel [3], [4]. Using completely different methods, Noonan and Zeilberger [6] extend the Goulden-Jackson Cluster Method [1] to find generating functions of word factors. In [2], [3], and [4], the authors find a myriad of results on generating functions related to word embeddings, and they subsequently use these generating functions to prove many results on Wilf-equivalence [7]. For instance, in [2] the authors define the embedding set of a word u to be $\mathcal{E}(u) = \{w \in \mathcal{P}^* \mid w \text{ embeds } u\}$, and then define that the words u and v are Wilf-equivalent, denoted as $u \sim v$, if $\mathcal{E}(u; t, x) = \mathcal{E}(v; t, x)$, where

$$\mathcal{E}(u; t, x) = \sum_{w \in \mathcal{E}(u)} \mathrm{wt}(w).$$

In this paper, we define a new notion of embedding which we call *generalized interval embedding*. To begin, suppose that we are given a word $u = u_1 u_2 \cdots u_m$

and a sequence of the same length as u, $A = (a_1, a_2, \ldots, a_m)$, where each $a_i \in \{0, 1, 2, 3, \ldots\} \cup \{\infty\}$. We say that $w = w_1 w_2 \cdots w_n$ *interval embeds* u^A, written as $u \leq_A w$, if there is a string, v, of $|u|$ consecutive letters in w such that for all $1 \leq i \leq |u|$, $v_i \in [u_i, u_i + a_i]$, where if $a_i = \infty$, we take this to mean that $v_i \in [u_i, \infty)$. Letting $a_i = 0$ for all i corresponds to interval embeddings which are factors, and letting $a_i = \infty$ for all i gives interval embeddings which are embeddings.

We define the interval embedding set of u^A to be

$$\mathcal{E}(u^A) = \{w \in \mathcal{P}^* \mid u \leq_A w\}.$$

Similarly, we define the avoiding set of u^A to be

$$\mathcal{A}(u^A) = \{w \in \mathcal{P}^* \mid u \not\leq_A w\},$$

and the suffix-embedding set of u^A to be

$$\mathcal{S}(u^A) = \{w \in \mathcal{P}^* \mid u \leq_A w \;\&\; u^A \text{ only interval-embeds into the suffix of } w\},$$

where we say that for any $1 \leq j \leq n$, $w_j w_{j+1} \cdots w_n$ (respectively, $w_1 w_2 \cdots w_j$) is a *suffix* (*respectively, prefix*) of w. We also define the same length-embedding set of u^A to be

$$\mathcal{L}(u^A) = \{w \in \mathcal{P}^* \mid u \leq_A w, \text{and } |u| = |w|\}.$$

We now define the corresponding weight generating functions to be

$$\begin{aligned}
\mathcal{E}(u^A; t, x) &= \sum_{w \in \mathcal{E}(u^A)} \mathrm{wt}(w), && (1)\\
\mathcal{A}(u^A; t, x) &= \sum_{w \in \mathcal{A}(u^A)} \mathrm{wt}(w), && (2)\\
\mathcal{S}(u^A; t, x) &= \sum_{w \in \mathcal{S}(u^A)} \mathrm{wt}(w), \text{and} && (3)\\
\mathcal{L}(u^A; t, x) &= \sum_{w \in \mathcal{L}(u^A)} \mathrm{wt}(w). && (4)
\end{aligned}$$

It is worth noting that there is a straightforward formula for $\mathcal{L}(u^A; t, x)$, which we provide here and use in Section 3.

Lemma 1. *Given a word $u = u_1 \cdots u_m$ and a sequence $A = (a_1, \ldots, a_m)$,*

$$\mathcal{L}(u^A; t, x) = t^m \prod_{i=1}^{m} \left(\sum_{j=0}^{a_j} x^{u_j + j} \right).$$

Proof. Let $w = w_1 w_2 \cdots w_m \in \mathcal{L}(u^A)$. Then we can compute $\mathcal{L}(u^A; t, x)$ by computing the possible weights of each w_i. Since $w_i \in \{u_i, u_i + 1, \ldots, u_i + a_i\}$, the sum of all possible weights for w_i is $t(x^{u_i} + x^{u_i+1} + \cdots + x^{u_i + a_i})$. Multiplying the sums of the possible weights for all letters of w yields the desired result. $\square$

It was shown in [2] that the weight generating function for all words in $\mathcal{P}^*$ is given by

$$P(t,x) = \sum_{w \in \mathbb{P}^*} \mathrm{wt}(w) = \frac{1-x}{1-x-tx}.$$

Since the sets $\mathcal{E}(u^A)$ and $\mathcal{A}(u^A)$ partition the set $\mathcal{P}^*$, we have that

$$\mathcal{E}(u^A;t,x) = \frac{1-x}{1-x-tx} - \mathcal{A}(u^A;t,x). \tag{5}$$

Further, every word $y \in \mathcal{E}(u^A)$ has a leftmost interval embedding of u^A. Accordingly, $y = y'z$, where $y' \in \mathcal{S}(u^A)$ and z is any word, giving that

$$\mathcal{E}(u^A;t,x) = \mathcal{S}(u^A;t,x)\frac{1-x}{1-x-tx}, \tag{6}$$

and

By Equations (6) and (5)

$$\mathcal{A}(u^A;t,x) = P(t,x) - \mathcal{E}(u^A;t,x) = P(t,x) - \mathcal{S}(u^A;t,x)P(t,x). \tag{7}$$

It follows from the results in [2] that $\mathcal{E}(u^A;t,x)$, $\mathcal{A}(u^A;t,x)$, and $\mathcal{S}(u^A;t,x)$ are each the difference of rational functions, and thus rational themselves, although this is not a main point of this paper. We say that two words, u and v, are *interval Wilf-equivalent* with respect to the sequences A and B, respectively, if $\mathcal{E}(u^A;t,x) = \mathcal{E}(v^B;t,x)$, and we denote this by $u^A \sim v^B$. It is helpful to note that by Equation (7), interval Wilf-equivalence is also shown if $\mathcal{S}(u^A;t,x) = \mathcal{S}(v^B;t,x)$ or $\mathcal{A}(u^A;t,x) = \mathcal{A}(v^B;t,x)$. One consequence of this definition is that any two words which are interval Wilf-equivalent must have equal weights, and we prove this fact here so that we may omit this condition in the statement of later theorems.

Lemma 2. *If $u^A \sim v^B$, then $wt(u) = wt(v)$.*

Proof. Assume $u^A \sim v^B$, that is, $\mathcal{E}(u^A;t,x) = \mathcal{E}(v^B;t,x)$. Given $w \in \mathcal{E}(u^A)$, $|w| \geq |u|$ and $\Sigma w \geq \Sigma u$. This gives that the lowest-order term in the power series expansion of $\mathcal{E}(u^A;t,x)$ is $\mathrm{wt}(u) = t^{|u|}x^{\Sigma u}$. Similarly, the lowest-order term in the power series expansion of $\mathcal{E}(v^B;t,x)$ is $\mathrm{wt}(v) = t^{|v|}x^{\Sigma v}$, and since these terms must be identical, we have that $\mathrm{wt}(u) = \mathrm{wt}(v)$. $\square$

For many of the results that follow, we are also interested in how many interval embeddings can "occupy the same space." To that end, we say that, for $0 \leq p \leq m-1$, u^A has *p-overlap* if the last p letters of u can share an embedding with the first p letters of u. In the special case where $p = 0$, we call u^A *non-overlapping*. For example, If $u = 121$, $v = 138$, and $A = (3,3,3)$, then u^A has both a 1-overlap and a 2-overlap; however, v^A is non-overlapping.

The outline of the paper is as follows. In Section 2, we give results for Wilf-equivalence in the most general of settings, that is, allowing our sequence A to be arbitrary. In Section 3, we discuss a special case of generalized interval embeddings, which we call k-embeddings.

2 Generalized interval embeddings

In this section we define some basic terms to help us tackle the problems in this generalized interval embeddings setting, and we discuss how to use these ideas to give us results on Wilf-equivalence. First, suppose u and v are words such that $|u| = |v| = m$ and $\Sigma u = \Sigma v$, and let $A = (a_1, a_2, \ldots, a_m)$ and $B = (b_1, b_2, \ldots, b_m)$ be two sequences made up of nonnegative integers and ∞'s, as defined in Section 1. Define $\overleftarrow{u} = u_m \cdots u_2 u_1$ and $\overleftarrow{A} = (a_m, \ldots, a_2, a_1)$ to be the *reverses* of u and A, respectively. Define u^+ to be the word obtained by increasing every letter of u by 1. We now give the following theorem, which is a generalized version of Lemma 4.1 in [2]. Here we provide the proofs of (1.) and the first half of (2.) in order to illustrate how one may prove other results in this setting, namely, by finding some weight-preserving bijection from one embedding (or avoiding or suffix-embedding) set to another. The proof second half of (2.) is omitted as it mirrors exactly the proof given below for the first half of (2.), and we refer the reader to [2] for the ideas behind the proof of (3.)

Theorem 3. *Suppose that u and v are words, A and B are sequences such that $u^A \sim v^B$. Then the following hold.*

1. $u^A \sim \overleftarrow{u}^{\overleftarrow{A}}$.

2. $(1u)^{A'} \sim (1v)^{B'}$ *and* $(u1)^{A''} \sim (v1)^{B''}$*, where for any word u and sequence A, $1u = 1u_1u_2 \cdots u_m$, $u1 = u_1u_2 \cdots u_m 1$, $A' = (\infty, a_1, a_2, \ldots, a_m)$, and $A'' = (a_1, a_2, \ldots, a_m, \infty)$.*

Proof. (1.) Suppose that w embeds u^A. Then $\phi : \mathcal{E}(u^A) \to \mathcal{E}(\overleftarrow{u}^{\overleftarrow{A}})$ given by $\phi(w) = \overleftarrow{w}$ is a weight-preserving bijection.

(2.) Consider $w \in \mathcal{A}((1u)^{A'})$. Then either u does not embed into w with respect to A or it embeds into a prefix of w, giving that

$$\mathcal{A}((1u)^{A'}) = \mathcal{A}(u^A) \uplus \{\overleftarrow{w} \mid w \in \mathcal{S}(\overleftarrow{u}^{\overleftarrow{A}})\},$$

where $\uplus$ represents a disjoint union of sets. A similar argument shows that

$$\mathcal{A}((1v)^{B'}) = \mathcal{A}(v^B) \uplus \{\overleftarrow{w} \mid w \in \mathcal{S}(\overleftarrow{v}^{\overleftarrow{B}})\}.$$

By (1.) and the fact that $u^A \sim v^B$, we have that $\mathcal{A}((1u)^{A'}) = \mathcal{A}((1v)^{B'})$. □

Define a *superset*, $[c, d] \subseteq \mathbb{N}$, of u^A to be a set such that for all $1 \leq i \leq m$, $[u_i, u_i + a_i] \subseteq [c, d]$, where we set $u_i + a_i = \infty$ whenever $a_i = \infty$, i.e., it may be the case that $[c, d] = [c, \infty)$. We now give the following conjecture, which is a generalization of Part (2.) of Theorem 3.

Conjecture 1. *Suppose that u and v are words, A and B are sequences such that $u^A \sim v^B$, and $[c, d]$ is s superset of both u^A and v^B. If $cu = cu_1u_2 \cdots u_m$, $uc = u_1u_2 \cdots u_m c$, $A' = (d - c, a_1, a_2, \ldots, a_m)$, and $A'' = (a_1, a_2, \ldots, a_m, d - c)$, with cv, vc, B', and B'' defined analogously, then $(cu)^{A'} \sim (cv)^{B'}$ and $(uc)^{A''} \sim (vc)^{B''}$.*

Kitaev et al [2], also pose a conjecture, given below, relating to the rearrangement of letters in two Wilf-equivalent words. This statement has come to be known as the *Rearrangment Conjecture.*

Conjecture 2. *If $u \sim v$, then u and v must be rearrangements of one another.*

In order to find the function $\mathcal{S}(u;t,x)$ for some arbitrary word u, the authors of [2] construct an automata which recognizes the words $w \in \mathcal{S}(u)$. One particular example (of many) given in that paper is that

$$123 \sim 321 \sim 231 \sim 132 \not\sim 213 \sim 312,$$

i.e., the converse of Conjecture 2 is false. However, in the form of the following theorem, we will show that there is an instance in which some form of rearrangement does force Wilf-equivalence. To begin, given $u = u_1u_2\cdots u_m$, $A = (a_1, a_2, \ldots, a_m)$, and a permutation $\sigma \in \mathcal{S}_m$, we define $\sigma(u) = u_{\sigma_1}u_{\sigma_2}\cdots u_{\sigma_m}$ and $\sigma(A) = (a_{\sigma_1}, a_{\sigma_2}, \ldots, a_{\sigma_m})$.

Theorem 4. *Suppose $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_m)$ are rearrangements of one another, and let u^A and v^B be non-overlapping with $wt(u) = wt(v)$. If there exists a permutation $\sigma \in \mathcal{S}_m$ such that $\sigma(A) = B$ and $\sigma(u)^{\sigma(A)}$ is non-overlapping, then $u^A \sim v^B$.*

In such a setting, $u^A \sim \sigma(u)^{\sigma(A)}$, and to see this, consider the word $w \in \mathcal{E}(u^A)$. Then either $w \in \mathcal{E}(u^A) \cap \mathcal{E}(\sigma(u)^{\sigma(A)})$ or $w \in \mathcal{E}(u^A) \setminus \mathcal{E}(\sigma(u)^{\sigma(A)})$. In the first case we do nothing, and in the second case, we can apply σ to each interval embedding of u^A and we will get a word, $w' \in \mathcal{E}(\sigma(u)^{\sigma(A)})$ such that $\mathrm{wt}(w) = \mathrm{wt}(w')$. Since u^A and $\sigma(u)^{\sigma(A)}$ are non-overlapping, no embeddings can be created or destroyed by this operation, so this map is also a bijection. As an example of this theorem, let $u = 711$ and $A = (0, 4, 2)$ while $v = 225$ and $B = (0, 2, 4)$. Then u^A and v^B are non-overlapping. Moreover, if $\sigma = 132$, we have that $\sigma(A) = B$ and $\sigma(u)^{\sigma(A)}$ is non-overlapping, and so $u^A \sim v^B$. Before proving this theorem, we first start with some simpler groundwork. To begin, assume $A = (a_1, a_2, \ldots, a_m) = B$, and let u^A and v^B be non-overlapping words with $\mathrm{wt}(u) = \mathrm{wt}(v)$. We define $\psi : \mathcal{A}(v^B) \to \mathcal{A}(u^A)$ as follows:

i. If $w \in \mathcal{A}(u^A) \cap \mathcal{A}(v^B)$, let $\psi(w) = w$.

ii. If $w \in \mathcal{A}(v^B) \setminus \mathcal{A}(u^A)$, first set $\Delta_{u,v} = (\delta_1, \delta_2, \ldots, \delta_m)$, where $\delta_i = v_i - u_i$. We note here that since $\Sigma u = \Sigma v$, $\delta_1 + \delta_2 + \cdots + \delta_m = 0$, and unless $u = v$, there exists some $\delta_j < 0$, a fact we will use to prove Lemma 5. Given a word $p = p_1p_2\cdots p_m$ we define $\Delta_{u,v}(p)$ to be the word whose i-th letter is $p_i + \delta_i$. We now define $\Gamma(w)$ to be the word obtained by once applying $\Delta_{u,v}$ to each subword $w' \in \mathcal{L}(u^A)$ of w. If $\Gamma(w) \in \mathcal{A}(u^A) \setminus \mathcal{A}(v^B)$, we set $\psi(w) = \Gamma(w)$. Otherwise, we keep applying Γ to our subsequent images until we arrive at some $\Gamma^j(w) \in \mathcal{A}(u^A) \setminus \mathcal{A}(v^B)$, and we set $\psi(w) = \Gamma^j(w)$. We will refer to the words $w, \Gamma(w), \Gamma^2(w), \ldots, \Gamma^j(w) = \psi(w)$ as the Γ-*sequence* of w.

As an example of part (ii.) of our algorithm, consider the words $u = 511$ and $v = 133$ with $A = B = (0, 1, 3)$. Then u^A and v^B are both non-overlapping, and $\Delta_{u,v} = (-4, 2, 2)$. The word $w = 515128523$ contains two interval embeddings of u^A, but w avoids v^B, and so we have that $w \in \mathcal{A}(v^B) \setminus \mathcal{A}(u^A)$. Thus, we apply $\Delta_{u,v}$ to all embeddings of u^A in w, and we keep applying $\Delta_{u,v}$ to embeddings in the subsequent images, until we arrive at a word in $\mathcal{A}(u^A) \setminus \mathcal{A}(v^B)$. We show this map here, where embeddings of u^A and are underlined in the top line and embeddings of v^B are overlined in the bottom line.

$$w = 51\underline{512}8\underline{523} \to \Gamma(w) = \underline{511}348145 \to \Gamma^2(w) = 133348145 = \psi(w)$$

$$w = 515128523 \to \Gamma(w) = 51\overline{134}8\overline{145} \to \Gamma^2(w) = \overline{133}158\overline{145} = \psi(w)$$

We now make two remarks before providing three technical lemmas which aid in the proof of Theorem 4.

Remark 1. *The leftmost letter of any word in our* Γ*-sequence can only be changed by increments of* δ_1 *at each step. Accordingly, if the first letter of any word in our* Γ*-sequence ever increases (decreases), then the first letter can never decrease (increase) in a later element of our* Γ*-sequence. A similar statement holds for the last letter of the words of our* Γ*-sequence.*

Remark 2. *As soon as* $\Delta_{u,v}$ *is applied to an interval embedding of* u^A*, that interval embedding is transformed into an interval embedding of* v^B*, although it may also still be an interval embedding of* u^A*. It may also be the case that this interval embedding of* u^A *shifts to the left or right. In the example above, such a left shift occurs.*

Lemma 5. *If* $p \in \mathcal{A}(v^B) \setminus \mathcal{A}(u^A)$ *such that* $|p| = |u|$*, then the* Γ*-sequence of* p *is finite.*

Proof. Let $p = p_1 p_2 \cdots p_m \in \mathcal{A}(v^B) \setminus \mathcal{A}(u^A)$ such that $|p| = |u|$, and let r be the minimum number of times we must apply Γ to p in order to no longer have an interval embedding of u^A. Note that such an r must exist, since there exists some $\delta_j < 0$ where $1 \le j \le m$, and so eventually we would be forced out of the interval $[u_j, u_j + a_j]$, even if $a_j = \infty$. Now suppose that the word we get from applying Γ r times to p is $q = q_1 q_2 \cdots q_m \in \mathcal{A}(u^A)$. Then for some $1 \le i \le m$, $q_i = p_i + r\delta_i = p_i + r(v_i - u_i) \notin [u_i, u_i + a_i]$, but since r is minimal, it must also be the case that $u_i \le p_i + (r-1)\delta_i = p_i + (r-1)(v_i - u_i) \le u_i + a_i$ for every $1 \le i \le m$. This gives that

$$u_i \le p_i + rv_i - ru_i - v_i + u_i \le u_i + a_i \Rightarrow v_i \le p_i + r(v_i - u_i) = q_i \le v_i + a_i,$$

and so $q \in \mathcal{L}(v^B)$. Thus, $q \in \mathcal{A}(u^A) \setminus \mathcal{A}(v^B)$, giving that $q = \Gamma^r(p) = \psi(p)$. $\square$

We now see that the only way in which this algorithm would fail to terminate is if, when applying Γ, interval embeddings of u^A could shift to the left or right ad infinitum. However, our next lemma shows that this cannot happen.

Lemma 6. *Given non-overlapping words u^A and v^B of length $m \geq 1$ and $w \in \mathcal{A}(v^B) \setminus \mathcal{A}(u^A)$, the Γ-sequence of w is finite.*

Proof. We proceed by induction on the length of the w.

Assume that $[u_1, u_1 + a_1]$ is finite, and let $P(n)$ be the statement that if we are given a word in $\mathcal{A}(v^B) \setminus \mathcal{A}(u^A)$ of length n, then the Γ-sequence of this word is finite. By Lemma 5, $P(m)$ is true, so assume that $P(k)$ is true for some $k \geq m$, and suppose we are given $w \in \mathcal{A}(v^B) \setminus \mathcal{A}(u^A)$ such that $|w| = k + 1$.

Since $[u_1, u_1 + a_1]$ is finite, Remark 1 gives us that the first letter of the words of the Γ-sequence of w can change only a finite number of times. That is, there exists $t \geq 0$ such that the first letter of $\Gamma^s(w)$ is fixed as y_1 for every $s \geq t$, and our Γ-sequence of w is

$$w, \Gamma(w), \Gamma^2(w), \ldots, \Gamma^t(w) = y_1 y_2 \cdots y_{k+1}, \Gamma^{t+1}(w) = y_1 y_2' \cdots y_{k+1}', \ldots.$$

If we let $y = y_2 y_3 \cdots y_{k+1}$, then since y_1 is not changing after applying Γ t times to w, our Γ-sequence of w can be written as

$$w, \Gamma(w), \ldots, \Gamma^t(w) = y_1 y, \Gamma^{t+1}(w) = y_1 \Gamma(y), \Gamma^{t+2}(w) = y_1 \Gamma^2(y), \ldots,$$

that is, after the t-th application of Γ to w, Γ is now only operating on the word y and its subsequent images. As $|y| = k$, our inductive hypothesis guarantees that the Γ-sequence of y, and consequently w, is finite.

Since u^A is non-overlapping, it must be the case that at least one of the intervals $[u_1, u_1 + a_1]$ and $[u_m, u_m + a_m]$ must be finite. Accordingly, if $[u_1, u_1 + a_1]$ is not finite, then $[u_m, u_m + a_m]$ is finite, and so we may use the same ideas above on the last letter of the words in the Γ-sequence of w, completing the proof. □

Lemma 7. *Suppose $A = (a_1, a_2, \ldots, a_m) = B$, and let u^A and v^B be non-overlapping with $wt(u) = wt(v)$. Then $u^A \sim v^B$.*

Proof. Pick $\hat{w} \in \mathcal{A}(u^A) \setminus \mathcal{A}(v^B)$. As v^B is non-overlapping, repeatedly applying $\Delta_{v,u} = -\Delta_{u,v}$ to embeddings of v^B in $\hat{w}$ and to all interval embeddings of v^B in the subsequent images, we would arrive back at a word $w \in \mathcal{A}(v^B) \setminus \mathcal{A}(u^A)$ such that $\psi(w) = \hat{w}$, that is, ψ is invertible, i.e., a bijection from $\mathcal{A}(v^B) \setminus \mathcal{A}(u^A)$ to $\mathcal{A}(u^A) \setminus \mathcal{A}(v^B)$. Since ψ is the identity map on $\mathcal{A}(u^A) \cap \mathcal{A}(v^B)$, ψ is a bijection from $\mathcal{A}(v^B)$ to $\mathcal{A}(u^A)$. Finally, $|w| = |\psi(w)|$, and since $\Sigma\delta_i = 0$, $\Sigma w = \Sigma\psi(w)$, so ψ is also a weight-preserving map. □

Since Wilf-equivalence is transitive and $u^A \sim \sigma(u)^{\sigma(A)}$, Theorem 4 now follows from first applying the appropriate permutation $\sigma \in \mathcal{S}_{|u|}$ to all interval embeddings of u^A in our word $w \in \mathcal{A}(v^B)$, and then applying Lemma 7 to obtain $\sigma(u)^{\sigma(A)} \sim v^B$.

Conjecture 3. *Suppose $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_m)$ are rearrangements of one another, and let u^A and v^B be non-overlapping with $wt(u) = wt(v)$. Then $u^A \sim v^B$.*

Conjecture 3 is the full generalization of Theorem 4, and all empirical evidence suggests that this conjecture holds true. However, the method used in the proof of Theorem 4 is no longer valid. In particular, when we try to find σ such that $\sigma(A) = B$, it may be the case that $\sigma(u)^{\sigma(A)}$ is no longer non-overlapping, and so when we try to apply ψ to words in $\mathcal{A}(v^B) \setminus \mathcal{A}(\sigma(u)^{\sigma(A)})$, ψ may not be well-defined.

3 k-embeddings

Returning to our original alphabet, $\mathcal{P} = \mathbb{N}$, let $k \geq 0$ and set $A = (k, k, \ldots, k)$. Here, we denote u^A as u^k, and we refer to embeddings in this case as *k-embeddings*. We denote the corresponding interval Wilf-equivalences of two words, u and v, as $u \sim_k v$. When $k = 0$ the k-embedding is equivalent to a factor, and when $k \to \infty$ the k-embedding is equivalent to an embedding; in fact, by choosing $k \in \mathbb{N}$, k-embeddings represent the "missing space" between factors and embeddings. For example, we will show in Theorem 8 that

$$\mathcal{E}(1^k; t, x) = \frac{tx(x^{k+1} - 1)(1 - x)}{(x - 1 + tx^{k+2})(1 - x - tx)}. \tag{8}$$

Letting $k = 0$ in Equation (8) gives the generating function for all words which contain the word $u = 1$ as a factor

$$G(1; t, x) = \frac{tx(x - 1)(1 - x)}{(x - 1 + tx^2)(1 - x - tx)}.$$

Similarly, let $k \to \infty$ in Equation (8) gives the generating function for all words which contain an embedding of $u = 1$, i.e., all words. Taking this limit we see that we do, in fact, get

$$\lim_{k\to\infty} \mathcal{E}(1^k; t, x) = \frac{tx}{1 - x - tx} = P(t, x).$$

Although it was defined in Section 1, we give a reformulation of the definition of p-overlap in the special case of k-embeddings. Here, given that $|u| = m$, we say that u^k has a *p-overlap* if there exists some $p \leq m$ such that for a length p suffix v of u, $|u_i - v_i| \leq k$ for all $i \in [1, p]$. A word u is *non-overlapping* if there is no p with this property.

3.1 Non-overlapping words

In this section we discuss those k-embeddings in the case where u^k is non-overlapping, and we begin by finding a generating function for words which k-embed such u^k.

Theorem 8. *Let $k \geq 0$ and let u^k be a non-overlapping word with $|u| = m$. Then,*

$$\mathcal{E}(u^k; t, x) = \frac{x^{\Sigma u} t^m (1 - x^{k+1})^m (1 - x)}{[(1 - x)^{m-1}(1 - x - tx) + x^{\Sigma u} t^m (1 - x^{k+1})^m](1 - x - tx)}.$$

Proof. Our strategy for finding $\mathcal{E}(u^k;t,x)$ involves first finding $\mathcal{S}(u^k;t,x)$. Take a word, v, that avoids u^k, then append any k-embedding of u^k of length $|u|$ to the right end to create an element of $\mathcal{S}(u^k)$. Because u^k is non-overlapping, we know that we have not created an earlier occurrence of u^k with a portion of our avoiding word and a portion of the k-embedding of u^k. Translating this into the world of generating functions gives that

$$\begin{aligned}
\mathcal{S}(u^k;t,x) &= \mathcal{A}(u^k;t,x)\mathcal{L}(u^k;t,x) \\
&= \mathcal{A}(u^k;t,x)\left[t^m \prod_{i=1}^{m}\left(\sum_{j=0}^{a_j} x^{u_j+j}\right)\right], \text{ by Lemma 1} \\
&= \mathcal{A}(u^k;t,x)t^m x^{\Sigma u}(1+x+\cdots+x^k)^m \\
&= \mathcal{A}(u^k;t,x)t^m x^{\Sigma u}\left(\frac{1-x^{k+1}}{1-x}\right)^m.
\end{aligned}$$

Recalling that

$$P(t,x) = \frac{1-x}{1-x-tx} \text{ and using Equations (6), (5), and (7),}$$

we obtain our generating function

$$\mathcal{E}(u^k;t,x) = \frac{x^{\Sigma u}t^m(1-x^{k+1})^m(1-x)}{[(1-x)^{m-1}(1-x-tx)+x^{\Sigma u}t^m(1-x^{k+1})^m](1-x-tx)}.$$

□

We see here that finding the generating function in Theorem 8 uses only the weight of the word u and the fact that u^k is non-overlapping, giving us the following corollary.

Corollary 9. *Let $k \geq 0$ and let u^k and v^k be non-overlapping words such that $wt(u) = wt(v)$. Then $u \sim_k v$.*

Indeed, this corollary gives that in the case of non-overlapping words and k-embeddings, Lemma 2 becomes an if and only if statement.

3.2 Connections to other sequences

Since interval embeddings of words of length one are always k-embeddings and such words are, by default, non-overlapping, we can use the ideas in Theorem 8 to find generating functions for the avoiding sets of words of length one.

Corollary 10. *Let $k \geq 0$ and suppose q is a word of length one. Then*

$$\mathcal{A}(q^k;t,x) = \frac{1-x}{1-x+t(x^q-x-x^{q+k+1})}. \tag{9}$$

Using Equation (9), we see that

$$\mathcal{A}(1^k;t,x) = \frac{1-x}{1-x-tx^{k+2}}.$$

By expanding this out as a power series for $k = 1$ and $k = 2$ and setting $t = 1$, i.e., disregarding the length of w, we obtain the following:

$$\begin{aligned}\mathcal{A}(1^1;1,x) &= 1 + x^3 + x^4 + x^5 + 2x^6 + 3x^7 + 4x^8 + 6x^9 + 9x^{10} + \cdots \\ \mathcal{A}(1^2;1,x) &= 1 + x^4 + x^5 + x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + \cdots .\end{aligned}$$

Searching in the OEIS for the sequences created from the coefficients of x^n in both of these polynomials, we see that they correspond to the sequences A000930 and A003269, respectively, and this leads us to the following result.

Theorem 11. *For $k \geq 1$, the coefficient of x^n in the power series expansion of $\mathcal{A}(1^k;1,x)$ is the number of words $w = w_1w_2\cdots w_{n-k-2}$ such that for all i, $w_i \in \{1, k+2\}$.*

Proof. Take a word w of length $n-k-2$ whose letters are elements of $\{1, k+2\}$, and form the word w' by appending the letter $k+2$ to the front of w. We will now form a word $z \in \mathcal{A}(1^k)$ such that $\Sigma z = n$ in the following way. First, starting from the left of w', sum from $k+2$ to the end of the largest string of consecutive 1's after this $k+2$, and let that be the first letter of z. Then sum the second $k+2$ with the largest string of 1's immediately after it, and let that be the second letter of z. We continue this algorithm until the word w' has been exhausted. Since every letter of z is at least $k+2$, $z \in \mathcal{A}(1^k)$. Moreover, by construction, $\Sigma w' = \Sigma z$, and $\Sigma w' = n - k - 2 + (k+2) = n$. Finally, this is a bijection, since this process is reversible. In particular, given a word $y \in \mathcal{A}(1^k)$ with $\Sigma y = n$, we replace any letter, y_i, of y which is greater than $k+2$ with the word $\ell_i = (k+2)11\cdots 1$ such $\Sigma \ell_i = y_i$, and we leave the letters $k+2$ untouched. This gives the a word v' which begins with the letter $k+2$, and we remove this first letter to get v, the preimage y. ☐

As an example of the bijection just described, consider the word $w = 4114141111$ with $k = 2$. Then $w' = 44114141111$, and so $z = (4)(4{+}1{+}1)(4{+}1)(4{+}1{+}1{+}1{+}1) = 4658 \in \mathcal{A}(1^2)$.

In general, words which avoid k-embeddings of single-letter words are related to pattern avoidance in binary words. To see this, let B_i denote the binary word of length $i+1$ which begins with a 0 and ends with i 1's, and let $B_{i,0}$ denote the word B_i with a 0 added after the i 1's. Then each word $w = w_1w_2\cdots w_t \in \mathbb{N}^*$ can be transformed into a unique binary word by replacing each w_i with B_i. This operation is invertible, and in fact, $\Sigma B_i = w_i$. Using this bijection, we get the following theorem, where the proof results from a direct application of this transformation.

Theorem 12. *For $q \in \mathbb{N}$ and $k \geq 1$, the coefficient of x^n in the power series expansion of $\mathcal{A}(q^k;1,x)$ is the number of binary words b with $\Sigma b = n$ that avoid exact macthes of all patterns in the set*

$$\Upsilon_{q,k} = \{B_{q,0}, B_{q+1,0}, \ldots, B_{q+k,0}\}.$$

This theorem is another example of how k-embeddings fill in some gap between factors and embeddings. It is in the vein of many results found in Miceli and Remmel [5], where this same bijection is used to find generating functions for words which contain certain sets of patterns as factors.

3.3 p-overlapping words

In this section we look at words which have p-overlap for some $p > 0$, and we begin with a generalization of Theorem 8.

Theorem 13. *Let $k \geq 0$ and suppose u is a word of length m such that there exists exactly one p such that u^k has a p-overlap. If $|u| \geq 2p$, then*

$$\begin{aligned}\mathcal{E}(u^k;t,x) &= \frac{1-x}{1-x-tx} \times \\ &\quad \frac{(1-x)(1-x^{k+1})^m t^m x^{\Sigma u}}{((1-x)^m + D(1-x-tx) + (1-x)t^m x^{\Sigma u}(1-x^{k+1})^m)},\end{aligned}$$

where

$$\begin{aligned}D &:= (1-x)^p t^{m-p} x^{\sum_{j=1}^{p} \max\{u_j, u_{m-p+j}\} + \sum_{r=1}^{m-2p} u_{p+r}} \times \\ &\quad (1-x^{k+1})^{m-2p} \prod_{l=1}^{m-2p} (1-x^{k+1-|u_l-u_{m-p+l}|}).\end{aligned}$$

Recall that in the proof of Theorem 8, we created the suffix-embedding set of u^k by taking any word that avoided u^k and appending any k-embedding of u^k of length $|u|$ to the right. Let's call the set of words created this way S_1. If we try the same strategy here, we encounter the difficulty that, because of the p-overlap that occurs, some of the words in S_1 would be in $\mathcal{E}(u^k;t,x)$ and not in $\mathcal{S}(u^k;t,x)$. For example, when $k = 2$ and $u = 141$, then u^k has one instance of p-overlap with $p = 1$. But, take the word 625, which contains no 2-embedding of 141, and append a 2-embedding of 141, such as 241, to obtain the word $w = 625241$. We see that $w \notin \mathcal{S}(141^2)$, even though $w \in \mathcal{E}(141^2)$. Thus the strategy from Theorem 8 has given us the set S_1 which contains all the words in $\mathcal{S}(141^2)$, but unfortunately some others as well. To fix this overcounting, we subtract the instances where there is an earlier embedding. Thus we want to subtract the set of words formed by taking any word that avoids u^k and appending any word w of length $2m-p$ with two embeddings of u^k. We will call this set S_2. (Note that since we know that there exists exactly one p such that u^k has p-overlap, these two embeddings in w must begin and end w). Taking $S_1 - S_2$ we have thus subtracted all words that were not in $\mathcal{S}(141^2)$. However, we have also subtracted some words which were never in the original set S_1! For example, with $u = 141$, $1625241 \in S_2$, because it is made of the word 16, which avoids 141 when $k = 2$, and the word $w = 25241$, which has two embeddings of u^2. But 1625241 is not in S_1, because the word 1625 does not avoid u^2. So we must add back in those words that are formed by taking any word that avoids u^k and appending any word of length $3m - 2p$ with three embeddings of u^k. We call this set S_3. This adds back in all the elements of $S_2 - S_1$ that were inappropriately removed. However, just as before, S_3 contains some words which are not in $\mathcal{S}(u^k)$, which we must subtract via a similarly defined set S_4. We continue this process in an inclusion-exclusion manner, finding that

$$\mathcal{S}(u^k) = S_1(u^k) - S_2(u^k) + S_3(u^k) - S_4(u^k) + \dots.$$

Then to find the generating function $\mathcal{S}(u^k; t, x)$, we need only find the generating functions for the S_i's and sum them with appropriate signs. Fortunately each S_i has a straightforward generating function. For example

$$S_1(u^k; t, x) = \mathcal{A}(u^k; t, x) t^m x^{\Sigma(u)} ([k]_x)^m,$$

and

$$\begin{aligned} S_2(u^k; t, x) &= \mathcal{A}(u^k; t, x) t^{2m-p} x^{\Sigma(u)} \left[\prod_{i=1}^{p} x^{\max\{u_i, u_{m-p-i}\}} \right] x^{u_{p+1}} \cdots x^{u_{m-p}} \\ &\quad \times \left[\prod_{i=1}^{m-2p} [k]_x [p - |u_i - u_{m-p+i}|]_x \right], \end{aligned}$$

where for any $x \in \mathbb{R}$ and $k \in \mathbb{N}$, $[k]_x := (1 + x + \cdots + x^k) = \dfrac{1 - x^{k+1}}{1 - x}$.

Continuing in this manner and factoring when possible, we see that $\mathcal{S}(u^k; t, x)$ has the form

$$\mathcal{A}(u^k; t, x) t^m x^{\Sigma(u)} ([k]_x)^m \quad \times$$
$$\sum_{i=0}^{\infty} \left(-t^{m-p} x^{\sum_{j=1}^{p} max\{u_j, u_{m-p+j}\} + \sum_{r=1}^{m-2p} u_{p+r}} \left[\prod_{l=1}^{m-2p} [k]_x [k - |u_l - u_{m-p+l}|]_x \right] \right)^i,$$

so that

$$\mathcal{S}(u^k; t, x) = \frac{\mathcal{A}(u^k; t, x) t^m x^{\Sigma(u)} (1 - x^{k+1})^m}{(1 - x)^m + D},$$

where D is defined in the statement of Theorem 13. Using Equations (6), (5), and (7) on this expression for $\mathcal{S}(u^k; t, x)$, the formula from Theorem 13 follows.

Whereas the last theorem dealt with words containing exactly one p-overlap, our next result pertains to the opposite case of p-overlapping. To begin, let $q \in \mathbb{N}$ and let q_ℓ denote the ℓ-length word comprised solely of the letter q. In this case, q_ℓ has a p-overlap for every $1 \le p \le \ell - 1$.

Theorem 14. *Let $k \ge 0$ and $\ell \in \mathbb{N}$. For any $q \in \mathbb{N}$,*

$$\mathcal{E}(q_\ell^k; t, x) = \frac{(1 - x)\overline{D}}{(1 - x - tx)(1 - x - tx + \overline{D})},$$

where $\overline{D} := t^\ell (x^q - x^{1+k+q}) + t^{1+k} (1 - x)^{-\ell} (x^q - x^{1+k+q})^\ell (x - x^q + x^{1+k+q})$.

Proof. We see that if $w = w_1 w_2 \cdots w_t \in \mathcal{S}(q_\ell^k)$, each of the last ℓ digits of w must be in $[q, q+k]$, and when $t > \ell$, it is also necessary for the letter $w_{t-\ell} \notin [q, q+k]$, since otherwise there will be a k-embedding of q_ℓ not contained in the last ℓ letters of w. Thus, if $|w| > \ell$, $w = uxv$, where $u \in \mathcal{A}(q_\ell^k)$, $x \in \mathbb{N} - [q, q+k]$, and $v \in \mathcal{S}(q_\ell^k)$, giving that

$$\begin{aligned}\mathcal{S}(q_\ell^k; t, x) &= \mathcal{A}(q_\ell^k; t, x) t^{k+1} (x^q [k]_x)^\ell \left(\frac{x - x^q}{1-x} + \frac{x^{q+k+1}}{1-x} \right) + \mathcal{L}(q_\ell^k; t, x) \\ &= \mathcal{A}(q_\ell^k; t, x) t^{k+1} (x^q [k]_x)^\ell \left(\frac{x - x^q}{1-x} + \frac{x^{q+k+1}}{1-x} \right) + t^\ell (x^q [k]_x)^\ell\end{aligned}$$

Using Equations (6), (5), and (7) to solve for $\mathcal{E}(q_\ell^k; t, x)$ yields the desired result. □

It is worth noting, as well, that in terms of Wilf equivalence, there is a fundamental, structural difference between non-overlapping words and words which have p-overlap for some $p > 0$. For example, $u = 181$ and $v = 262$ have the same weights and both have a single p-overlap for $p = 1$, but $181 \not\sim_1 262$. Thus, Corollary 9 no longer holds even in the simplest cases of non-zero p-overlaps. Moreover, we have the following theorem.

Theorem 15. *Let $k \geq 0$, u be a nonoverlapping word, and v be a word with p-overlap for some $p > 0$. Then $u \not\sim_k v$.*

Proof. We again consider $\mathcal{S}(u^k; t, x) = \mathcal{A}(u^k; t, x)\mathcal{L}(u^k; t, x)$, ss in the proof of Theorem 8.

Let $\tilde{\mathcal{S}}(v^k)$ be the set of all words w which have a k-embedding of v in the last $|v|$ letters of w and which also have at least one other earlier k-embedding of v in the last $2|v| - 1$ letters of w. Define $\tilde{\mathcal{S}}(v^k; t, x)$ to be the weight-generating function for $\tilde{\mathcal{S}}(v^k)$. Note that $\tilde{\mathcal{S}}(v^k)$ is non-empty since v is a p-overlapping word, so that $\tilde{\mathcal{S}}(v^k; t, x) \neq 0$. With these definitions, we have that $\mathcal{S}(v^k) = \mathcal{A}(v^k)\mathcal{L}(v^k) - \tilde{\mathcal{S}}(v^k)$, giving that $\mathcal{S}(v^k; t, x) = \mathcal{A}(v^k; t, x)\mathcal{L}(v^k; t, x) - \tilde{\mathcal{S}}(v^k; t, x)$.

Now, assume for the sake of contradiction that $u \sim_k v$. Then $\mathcal{A}(u^k; t, x) = \mathcal{A}(v^k; t, x)$ and by Lemma 2, $\mathcal{L}(u^k; t, x) = \mathcal{L}(v^k; t, x)$. This gives that

$$\begin{aligned}\mathcal{S}(u^k; t, x) &= \mathcal{A}(u^k; t, x)\mathcal{L}(u^k; t, x) \\ &= \mathcal{A}(v^k; t, x)\mathcal{L}(v^k; t, x) \\ &\neq \mathcal{A}(v^k; t, x)\mathcal{L}(v^k; t, x) - \tilde{\mathcal{S}}(v^k; t, x) \\ &= \mathcal{S}(v^k; t, x).\end{aligned}$$

□

3.4 δ^k-disjoint words and conjectures

Finally, we wish to provide an example of one more type of word which has some interesting properties. In this section, because of the inadequacy of examples in which the words consist only of single-digit letters, we must use some words which contains double-digit letters. To that end, we will write words with commas between the letters, and in this way, we can differentiate between the letter one followed by the letter three (1,3) and the letter thirteen (13).

Let $u = u_1, u_2, \ldots, u_m$ be a nonempty word in $\mathbb{N}^*$ and let $\delta \in \mathbb{N}$ such that δ divides $|u| = m$. Given $i \in \{1, 2, \ldots, \delta\}$, we define the sequence

$$U_i = (u_i, u_{\delta+i}, u_{2\delta+i}, \ldots, u_{m-\delta+i}),$$

and we define the collection of sequences $U_1, U_2, \ldots, U_\delta$ to be the δ-*sequences* of u. Given $k > 0$, we say that u^k is δ^k-*disjoint* if for all $i, j \in \{1, \ldots, \delta\}$ with $i \neq j$

1. $|x - y| \leq k$ whenever $x, y \in U_i$, and
2. $|x - y| > k$ whenever $x \in U_i$ and $y \in U_j$.

To put this definition in terms of some of our embedding sets, suppose we create a word v_i of length m/δ by using each letter of U_i exactly once, and suppose we similarly form a word v_j using the letters of U_j. Then u^k is δ^k-disjoint if $\mathcal{L}(v_i^k) \cap \mathcal{L}(v_j^k) = \emptyset$ for every $i \neq j$. As an example, consider the word $u = 1, 6, 11, 2, 7, 12, 3, 8, 13$. For $\delta = 3$, the δ-sequences of u are $U_1 = (1, 2, 3)$, $U_2 = (6, 7, 8)$, and $U_3 = (11, 12, 13)$, and we also see that u^2 is 3^2-disjoint. Similarly, if $\delta = 2$ and $v = 7, 1, 6, 2, 7, 3$, then our δ-sequences are $V_1 = (7, 6, 7)$ and $V_2 = (1, 2, 3)$; moreover, v^2 and v^3 are 2^2 and 2^3-disjoint, respectively.

Given a sequence $X = (x_1, x_2, \ldots, x_m)$ and $k \in \{0, 1, \ldots, m-1\}$, we define the k-*shift* of X to be the sequence $X' = (x'_1, x'_2, \ldots x'_m)$ where $x_i = x'_{i+k \bmod m}$, that is, X' is obtained by cycling all elements of X to the right by k spaces. For example, the 3-shift of $X = (1, 2, 3, 4, 5, 6, 7)$ is $X' = (5, 6, 7, 1, 2, 3, 4)$. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_\delta \in \{0, 1, 2, \ldots, \frac{m}{\delta} - 1\}^*$. We then define the σ-*shift of* u, $\sigma(u)$, to be the word constructed by replacing the letters of u corresponding to the δ-sequences $U_1, U_2, \ldots, U_\delta$ with those that correspond to $U'_1, U'_2, \ldots, U'_\delta$, where U'_i is obtained from a σ_i-shift of U_i. As an example, again consider the 3^2-disjoint word $u = 1, 6, 11, 2, 7, 12, 3, 8, 13$. Letting $\sigma = 102$, we get that

$$U_1 = (u_1, u_4, u_7) = (1, 2, 3) \mapsto U'_1 = (u'_1, u'_4, u'_7) = (u_7, u_1, u_4) = (3, 1, 2),$$

and similarly, $U'_2 = (6, 7, 8)$ and $U'_3 = (12, 13, 11)$, so that

$$\sigma(u) = 3, 6, 12, 1, 7, 13, 2, 8, 11.$$

Using this notion of shifts in δ-disjoint words, and based on computational data, we arrive at the following conjecture.

Conjecture 4. *If u is a word with no repeated letters and $k \geq 0$ such that u^k is δ^k-disjoint, then if τ and σ are shifts of u and τ is a rearrangement of the letters of σ, then $\sigma(u) \sim_k \tau(u)$.*

As an example, consider again the word $u = 1, 6, 11, 2, 7, 12, 3, 8, 13$, which we saw above is 3^2-disjoint. Computational results show that if $\sigma, \tau \in \{100, 010, 001\}$, then $\sigma(u) \sim_2 \tau(u)$. However, if $\sigma \in \{100, 010, 001\}$ and $\tau \in \{0, 1, 2\}^3 \setminus \{100, 010, 001\}$, then $\sigma(u) \not\sim_2 \tau(u)$.

We also did a great deal of investigation into the behavior of δ^k-disjoint words in which one δ-sequence contains repeated letters, and this leads us to the following conjecture.

Conjecture 5. *Suppose u is a word and $k \geq 0$ such that u^k is δ^k-disjoint. Further suppose that U_i is the only δ-sequence of u that has repeated letters. Then, if τ and σ are shifts of u and $\sigma_1\sigma_2 \cdots \sigma_{i-1}\sigma_{i+1} \cdots \sigma_\delta$ is a rearrangement of $\tau_1\tau_2 \cdots \tau_{i-1}\tau_{i+1} \cdots \tau_\delta$ and σ_i, τ_i are shifts such that U_i remains unchanged, then $\sigma(u) \sim_k \tau(u)$.*

Here, consider the word $u = 1, 5, 11, 2, 7, 12, 1, 6, 13, 2, 8, 14$, which is 3^2-disjoint. In this case, our δ-sequences are $U_1 = (1, 2, 1, 2)$, $U_2 = (5, 7, 6, 8)$, and $U_3 = (11, 12, 13, 14)$, so that if U_1 is 2-shifted, then it is the same as leaving U_1 unshifted. Our conjecture then states, for example, that

$$210(u) \sim_2 201(u) \sim_2 010(u) \sim_2 001(u),$$

a result which may be computationally verified.

4 Further work and acknowledgements

Finding the generating function for u^A with more than one p-overlap remains an open question for general sequences A, and the implicit inclusion/exclusion arguments needed for such problems may show themselves to be messy to give explicit combinatorial formulae. Even in the special case of k-embeddings, finding generating functions and proving results on Wilf equivalence is difficult, and more firepower may be necessary. These issues are the subject of future work, where variations of the i Goulden-Jackson Cluster Method [1] will be employed.

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