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# Cofinal Types and Bounding Numbers: A Literature Review

Spencer Chapman
*Trinity University*

**Cofinal Types and Bounding Numbers: A Literature Review**

**Spencer Chapman**

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**Dr. Julio Roberto Hasfura-Buenaga**

THESIS ADVISOR

**Dr. Eduardo Cabral Balreira**

DEPARTMENT CHAIR

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# Cofinal Types and Bounding Numbers: A Literature Review

Spencer Chapman

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## 1 Preface

This honor's thesis will be divided into two parts. The introductory section will be essentially a literature review surrounding the work on cofinal types introduced by Tukey, with an emphasis on their behavior under additional set-theoretic axioms as discussed heavily by Todorcevic. It will also include a section on bounding numbers and their role in combinatorial set theory, directed towards their applications to cofinal types. The second part of this thesis will include an introduction to Cohen forcing, and will include some novel results on the classification of cofinal types in generic models created from Cohen forcing. The thesis will then conclude with some conjectures and ideas for future work. The material presented in this introductory section is created following the lecture notes and advice of Dr. Osvaldo Guzmán at the Centro de Ciencias Matemáticas (CCM), UNAM Campus Morelia. We would like to thank Dr. Guzmán for presenting this research idea to us, as well as his continued assistance on the project. The introduction section of this thesis should be considered as a translation and elaboration of his expository on the subject.

## 2 Introduction

Let $X$ be a set and $\mathcal{I} \subseteq \mathcal{P}(X)$. We say $\mathcal{I}$ is an *ideal* if

1. $\emptyset \in \mathcal{I}$,
2. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$,
3. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

We will typically assert that $X \notin \mathcal{I}$. In this case, $\mathcal{I}$ is known as a *proper ideal*. Moreover, we observe that $[X]^{<\omega} \subseteq \mathcal{I}$. We denote $\mathcal{I}^\dagger = \{A \subseteq X : A \notin \mathcal{I}\}$. Let $X$ and $Y$ be sets, with $\mathcal{I}$ an ideal in $X$ and $\mathcal{J}$ an ideal in $Y$, and a map $f : X \to Y$. We say that a map

$$f : (X, \mathcal{I}) \to (Y, \mathcal{J})$$

is *Katetov* if for all $A \in \mathcal{J}$, we have that $f^{-1}(A) \in \mathcal{I}$.
We may then define the Katetov ordering of ideals, and say $\mathcal{J} \leq_K \mathcal{I}$ if there exists a map $g : (X, \mathcal{I}) \to (Y, \mathcal{J})$ that is Katetov. We also say $\mathcal{I}$ and $\mathcal{J}$ are Katetov equivalent (denoted $\mathcal{I} =_K \mathcal{J}$) if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$.

**Lemma 1.** *Let $\mathcal{I}$ be an ideal in $X$ and $\mathcal{J}$ an ideal in $Y$, with a map $f : X \to Y$. Then the following are equivalent:*

1. *$f : (X, \mathcal{I}) \to (Y, \mathcal{J})$ is Katetov.*
2. *For all $B \subseteq X$, if $B \in \mathcal{I}^\dagger$ then $F[B] \in \mathcal{J}^\dagger$.*

*Proof.* Suppose towards contrapositive, that there exists a $B \in \mathcal{I}^\dagger$ such that $F[B] \in \mathcal{J}$. Since $B \subseteq f^{-1}(f[B]) \in \mathcal{I}^\dagger$, we have that $f$ is not Katetov. Conversely, if $f$ is not Katetov, then there exists an $A \in \mathcal{J}$ such that $f^{-1}(A) \in \mathcal{I}^\dagger$. Since $f[f^{-1}(A)] \subseteq A$ and $\mathcal{J}$ is an ideal, $f[f^{-1}(A)] \in \mathcal{J}$. □

Let $(D, \leq)$ be a partially ordered set. We say that $D$ is a *directed* set if for all $a, b \in D$, there exists a $c \in D$ such that $a \leq c$ and $b \leq c$.
For a directed set, it will be useful to define the following sets:

1. $\mathrm{bnd}(D) = \{B \subseteq D : B \text{ is bounded}\}$
2. $\mathrm{ncf}(D) = \{B \subseteq D : B \text{ is not cofinal}\}$.

It is clear that $\mathrm{bnd}(D), \mathrm{ncf}(D) \subseteq \mathcal{P}(D)$. The empty set is trivially bounded and not cofinal. Moreover, the union of any two bounded/not-cofinal sets is itself bounded/not-cofinal, as are their subsets.

**Lemma 2.** *If $D$ is a directed set, then $\mathrm{bnd}(D)$ and $\mathrm{ncf}(D)$ are ideals on $D$.* □

The following propositions will relate the Katetov ordering of ideals to the Tukey ordering presented in [4].

Let $D$ and $E$ be directed sets. We denote the *Tukey ordering* on the class of cofinal sets as $\leq_T$, defined by $D \leq_T E$ if $(D, \mathrm{ncf}(D)) \leq_K (E, \mathrm{ncf}(E))$. We say that $D$ and $E$ are *Tukey equivalent* (denoted $D =_T E$) if $D \leq_T E$ and $E \leq_T D$.

**Proposition 3.** *Let $D$ and $E$ be directed sets. Then the following are equivalent:*

1. *$D \leq_T E$,*
2. *$(D, \mathrm{ncf}(D)) \leq_K (E, \mathrm{ncf}(E))$,*
3. *$(E, \mathrm{bnd}(E)) \leq_K (D, \mathrm{bnd}(D))$,*
4. *There exists maps $f : E \to D$ and $g : D \to E$ such that for all $d \in D$ and $e \in E$, if $g(d) \leq e$ then $d \leq f(e)$.*

We see that (*1*) and (*2*) are equivalent by the above definition. Moreover, (*4*) is the standard approach to Tukey ordering, presented as convergent maps in [4].

*Proof.* (*2* $\implies$ *4*): Let $f : (E, \mathrm{ncf}(E)) \to (D, \mathrm{ncf}(D))$ be Katetov. We then define $g : D \to E$ as follows: given $p \in D$, let

$$X_d = \{a \in D : d \not\leq a\}.$$

We see that $X_p$ is not cofinal, hence $f^{-1}(X_p)$ is not cofinal. We then choose $g(d)$ such that for all $e \in f^{-1}(X_d)$, $g(d) \not\leq e$. Equivalently, $d \not\leq f(e)$ implies that $g(d) \not\leq e$. Thus by contrapositive, $g(d) \leq e$ implies that $d \leq f(e)$.

(*3* $\implies$ *4*): Similarly, let $g : (D, \mathrm{bnd}(D)) \to (E, \mathrm{bnd}(E))$ be Katetov. We define $f : E \to D$ as follows: For $e \in E$ let

$$Y_e = \{a \in E : a \leq e\}.$$

Since $Y_e$ is bounded, $g^{-1}(Y_e)$ is bounded. We then pick $f(e)$ such that for any $d \in g^{-1}(Y_q)$, $d \leq f(e)$. Therefore, for all $d \in D$ and $e \in E$, $d \in g^{-1}(Y_e)$ implies that $d \leq f(e)$, and thus $g(d) \leq e$ implies that $d \leq f(e)$.
(*4* $\implies$ *2* and *3*). Let $f$ and $g$ be maps as described. We will show that $f : (E, \mathrm{ncf}(E)) \to (D, \mathrm{ncf}(D))$ are Katetov. Let $C \subseteq E$ be cofinal. Let $d \in D$. Then, $g(d) \in E$. Since $C$ is cofinal, there exists an $e \in C$ such that $g(d) \leq e$, hence $d \leq f(e)$, so $f(C)$ is cofinal. Similarly, we show that $g : (D, \mathrm{bnd}(D)) \to (E, \mathrm{bnd}(E))$ is Katetov. Let $A \subseteq D$ be unbounded. Suppose there exists an $e \in E$ such that if $g(d) \leq e$ for all $d \in A$, then $d \leq f(e)$ for all $d \in A$. However, this implies that $A$ is bounded, a contradiction. □

**Corollary 4.** *Let $D$ and $E$ be directed sets. Then the following are equivalent:*

1. $D \leq_T E$
2. *There exists a map $f : E \to D$ that maps cofinal subsets to cofinal subsets.*
3. *There exists a map $g : D \to E$ that maps unbounded subsets to unbounded subsets. Such a map is called a Tukey map.*
4. *There exists maps $f : E \to D$ and $g : D \to E$ such that for all $d \in D$ and $e \in E$, if $g(d) \leq e$ then $d \leq f(e)$.* □

**Lemma 5.** *Let $D$ be a directed set. If $A \subseteq D$ is cofinal, then $A =_T D$.*

*Proof.* The inclusion map $\mathrm{Id} : A \to D$ is a cofinal map, and so $A \leq_T D$. Moreover, Id is Tukey, so $D \leq A$. □

**Theorem 6** (Tukey)**.** *Let $D$ and $E$ be directed sets. Then $D =_T E$ if and only if there exists a directed set $\mathbb{D}$ such that $D$ and $E$ are both isomorphic to cofinal subsets of $\mathbb{D}$.*

*Proof.* We see that the converse follows from the above lemma. Suppose that $D =_T E$. Then there exists maps $f : D \to E$ and $g : E \to D$ such that if $d \in D$ and $e \in E$, $\geq_D g(e)$ implies that $f(d) \geq_E e$ and $e \geq_E f(d)$ implies that $g(e) \geq_D d$. Assuming that $D$ and $E$ are disjoint, let $X = D \cup E$. We then define

$\leq_X$ as follows: $\leq_X$ restricted to $D$ is $\leq_D$, and $\leq_X$ restricted to $E$ is $\leq_E$. For $d \in D$ and $e \in E$, we say $e \leq_X d$ iff $g(e') \leq_D d$ for some $e' \geq_E e$ and $d \leq_X e$ iff $f(d') \leq_E e$ for some $d' \geq_D d$. It may be shown that $X$ is a quasi-ordering. Then, taking the family of equivalence classes $\{[x] : x \in X\}$ with respect to the equivalence relation $\leq_X$, and denote this set by $\bar{X}$. We then have that $E$ and $D$ are isomorphic to cofinal subsets of $\bar{X}$ by sending $[e]$ to $e$ and $[d]$ to $d$ for each $e \in E$ and $d \in D$. □

With the introduction of the Tukey ordering completed, we may look at how various directed sets compare to eachother.

**Proposition 7.** *Let $D$ be a directed set. Let $1$ represent the singleton directed set with the trivial ordering. Then, $1 \leq_T D$. Moreover, $D =_T 1$ if and only if $D$ has a maximal element.*

*Proof.* Observe that $f : D \to 1$ is a cofinal map. Moreover, if $g : 1 \to D$ is a cofinal map, then the only point in the image of $g$ is the maximum of $D$. □

The product of directed sets is itself directed with the obvious ordering. In particular, the product of directed sets is not only higher in the Tukey ordering, but it is the least upper bound of its products. This is proved in [4] for any finite product, and we present the proof for the product of two directed sets for the sake of completeness.

**Proposition 8.** *Let $D$ and $E$ be directed sets. Then $D, E \leq_T D \times E$. Moreover, if $X$ is directed and $D, E \leq_T X$, then $D \times E \leq_T X$.*

*Proof.* First, fix a $e_0 \in E$. We define the map $f : D \to D \times E$ where $f(d) = (d, e_0)$. As this is an unbounded map, $D \leq_T D \times E$. The same may be shwon for $E$. Moreover, let $f : D \to X$ and $g : E \to X$ be Tukey. We define $h : D \times E \to X$ such that for every $(d, e) \in D \times E$, we have that $f(d), g(e) \leq h(d, e)$. Thus, $h$ is Tukey. □

The next section will discuss what the maximal directed set looks for like in the family of directed sets of size at most some cardinal. Let $\kappa$ be a cardinal. We denote $[\kappa]^{<\omega}$ to be the set of finite subsets of $\kappa$. We note that $[\kappa]^{<\omega}$ is a directed set when ordered by inclusion.

**Proposition 9.** *Let $D$ be a directed set. If $|D| \leq \kappa$, then $D \leq_T [\kappa]^{<\omega}$.*

*Proof.* We will enumerate $D$, possibly with duplicates, by $D = \{d_\alpha : \alpha \in \kappa\}$. We define the map $f : [\kappa]^{<\omega} \to D$ such that $d_{\alpha_1}, \ldots, d_{\alpha_n} \leq f(S)$ where $S = \{\alpha_1, \ldots, \alpha_n\}$. We see that $f$ is a cofinal map. Indeed, let $C \subseteq [\kappa]^{<\omega}$ be cofinal and $d_\alpha \in D$. As $C$ is cofinal, there exists an $S \in C$ such that $\{\alpha\} \subseteq S$. It then follows that $d_\alpha \leq f(S)$. □

It has not been discussed yet, but it may be easily shown that $=_T$ is an equivalence relation on the class of directed sets. We call the equivalence classes under the relation $=_T$ the *cofinal types*, or the *Tukey types*. We have shown that $[\kappa]^{<\omega}$ is the largest cofinal type on the class of directed sets of cardinality at most $\kappa$. The rest of this section will be showcasing the work by Tukey and Todorcevic on classifying the other cofinal types.

**Proposition 10.** *Let $D$ be a directed set. Then $[\kappa]^{<\omega} \leq_T D$ if and only if there exists a subset $\{d_\alpha : \alpha \in \kappa\} \subseteq D$ such that for all $S \in [\kappa]^{<\omega}$, the set $\{d_\alpha : \alpha \in S\}$ is unbounded in $D$.*

*Proof.* For the forward direction, let $f : [\kappa]^{<\omega} \to D$ be Tukey. We see that $\mathrm{ran}(f)$ satisfies the above. Conversely, define $f : [\kappa]^{<\omega} \to D$ such that for all $S \in [\kappa]^{<\omega}$,

$$d_{\alpha_1}, \ldots, d_{\alpha_n} \leq f(S),$$

where $S = \{\alpha_1, \ldots, \alpha_n\}$. Thus, $f$ is Tukey. □

**Corollary 11.** *Let $D$ be a directed set of size $\kappa$. Then $D =_T [\kappa]^{<\omega}$ if and only if there exists a set $\{d_\alpha : \alpha \in \kappa\} \subseteq D$ such that for all $S \in [\kappa]^{<\omega}$, the set $\{d_\alpha : \alpha \in S\}$ is unbounded.* □

**Corollary 12.** *Let $D$ be a directed set. Then the following are equivalent:*

1. *$\omega \leq_T D$,*
2. *There exists $W \in [D]^\omega$ such that every infinite subset of $W$ is unbounded,*
3. *There exists $W \in [D]^\omega$ that is unbounded.*

*Proof.* We see that (*1*) is equivalent to (*2*) simply as a property of $[\omega]^{<\omega}$. Moreover, (*3*) follows trivially from (*2*). Let $W = \{w_n : n \in \omega\} \subseteq D$ be unbounded. Since $D$ is directed, we may define recursively $B = \{b_n : n \in \omega\}$ such that if $n < m$ then $w_n < b_m$. □

So, we have shown that there are maximal and minimal directed sets in the Tukey ordering. The rest of this section will discuss the work on classifying directed sets of cardinality at most $\omega_1$. Firstly, we observe that there are only ever two countable cofinal types.

**Proposition 13.** *Let $D$ be a directed set of size $\omega$. Then the following are true:*

1. *$D =_T 1$ or $D =_T \omega$,*
2. *$D =_T 1$ if and only if $D$ has a maximal element,*
3. *$D =_T \omega$ if and only if $D$ has no maximal element.*

*Proof.* If $D$ has no maximal element, we can find a cofinal subset of $D$ isomorphic to $\omega$. □

Our second observation is that there exists cofinal types that are not Tukey comparable. Namely, $\omega$ and $\omega_1$.

**Proposition 14.** *Let $D$ be a directed set. Then $\omega_1 \leq_T D$ if and only if there exists an $S \in [D]^{\omega_1}$ such that every uncountable subset of $S$ is unbounded.* □

**Proposition 15.** *It is true that* $\omega \not\leq_T \omega_1$ *and that* $\omega_1 \leq_T \omega$.

*Proof.* Every countable subset of $\omega_1$ is bounded. Moreover, $\omega$ has no uncountable subsets. $\square$

We will now discuss where $\omega \times \omega_1$ sits in the Tukey ordering relative to the other cofinal types thus far. Let $D$ be a directed set and let $B \subseteq D$. We say that $B$ is $\omega$-*bounded* if every countably infinite subset of $B$ is bounded in $D$.

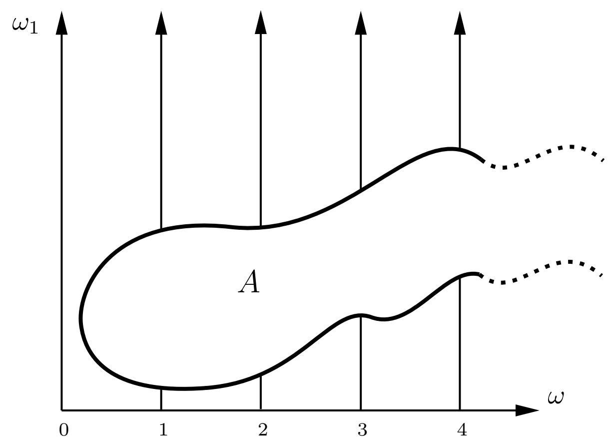


Figure 1: An $\omega$-bounded subset of $\omega \times \omega_1$.

Note that every column is $\omega$-bounded. If $A$ intersects infinitely many columns, then $A$ is $\omega$-bounded in $\omega \times \omega_1$. The concept of $\omega$-boundedness allows us to talk about $\omega \times \omega_1$ in the Tukey ordering.

**Proposition 16.** *It is true that* $\omega \times \omega_1 <_T [\omega_1]^{<\omega}$.

*Proof.* We have shown that $\omega \times \omega_1 \leq_T [\omega_1]^{<\omega}$. However, since $\omega \times \omega_1$ is $\omega$-bounded, any uncountable subset contains a countable, bounded subset. Thus the above inequality is strict. $\square$

**Proposition 17.** *Let $D$ be a directed set of size $\omega_1$. Then $D \leq_T \omega \times \omega_1$ if and only if $D$ is the countable union of $\omega$-bounded sets.*

*Proof.* Let $f : D \to \omega \times \omega_1$ be Tukey. We recall that in $\omega \times \omega_1$, each column $\{n\} \times \omega_1$ is $\omega$-bounded. So, let $B_n = f^{-1}(\{n\} \times \omega_1)$, and so $D = \bigcup_{n<\omega} B_n$. Since $f$ is Tukey, each $B_n$ is also $\omega$-bounded. Conversely, let $D = \bigcup_{n<\omega} B_n$, where each $B_n$ is $\omega$-bounded. For simplicty, we may assume that each $B_n$ is disjoint. Then, $|B_n| \leq |D| = \omega_1$ for each $n \in \omega$. That is, for each $n$ there exists a function $f_n : B_n \to \omega_1$ that is injective. We then define $f : D \to \omega \times \omega_1$ such that

$$f(d) = (n, f_n(d))$$

where $d \in B_n$. We see that $f$ is Tukey. Indeed, let $X \subseteq D$ be unbounded. We will show that $f(X)$ is unbounded. We note that if $X \cap B_n \neq \emptyset$, for infinitely many $n$, then we are done. Suppose that is not the case, then there exists an $n$ such that $Y = X \cap B_n$ is unbounded. As $B_n$ is $\omega$-bounded, $Y$ must be uncountable, and therefore $F(Y)$ is unbounded. □

**Proposition 18.** *Let $D$ be a directed set of size $\omega_1$. Then $D$ is Tukey comparable to $\omega \times \omega_1$. That is, $\omega \times \omega_1 \leq_T D$ and/or $D \leq_T \omega \times \omega_1$.*

*Proof.* If $D$ is $\omega$-bounded, then $D \leq_T \omega \times \omega_1$. Instead, suppose that $D$ is not $\omega$-bounded. Then we will show that $\omega \times \omega_1 \leq_T D$. Indeed, it suffices to show that $\omega \leq_T D$ and $\omega_1 \leq_T D$. Since $D$ is not $\omega$-bounded, it must contain an unbounded countable subset. So, $\omega_1 \leq_T D$. Now, let $f : D \to \omega_1$ be a bijection. We see that $f$ is a cofinal map. Indeed, it suffices to show that $D$ has no countable cofinal subsets. This must be true, otherwise $D =_T \omega <_T \omega \times \omega_1$ by, implying that $D$ is $\omega$-bounded. Thus, $D \leq_T \omega_1 \times \omega$. □

## 2.1 A Sixth Cofinal Type

The following proposition was presented by Todorcevic in [4] and follows from our work above. We will state it here for the sake of completeness.

**Proposition 19** (Todorcevic)**.** *Let $D$ be a directed set of size at most $\omega_1$. Then either $D =_T 1$, or $D =_T \omega$, or $D =_T \omega_1$, or $\omega \times \omega_1 \leq_T D \leq_T [\omega_1]^{<\omega}$.*

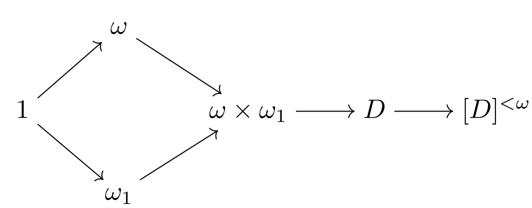


Figure 2: A diagram of the possible cofinal types of size at most $\omega_1$.

The motivation for Todorcevic's work, and for ours are inspired by the following questions:

1. Is it provable from some extension of ZFC that there are more than five cofinal types of directed sets of size at most $\omega_1$?

2. Under what assumptions can we have exclusively these five cofinal types?

3. How does the classification of cofinal types in a generic extension compare to that in a ground model?

The first two questions were answered positively by Todorcevic in [5] and [4]. Namely, Martin's Maximum implies that the above five cofinal types are the only ones of size at most $\omega_1$. However, Todorcevic also showed that there are many cofinal types of size continuum, and consequently many cofinal types of size $\omega_1$ under the Continuum Hypothesis. This section will now cover Todorcevic's work on answering these first two questions. The following theorem of Todorcevic shows answers the second question. The proof is beyond the scope of this paper, and may be read in either [4] or [5].

**Theorem 20** (Todorcevic)**.** *If* $\mathfrak{mm} > \omega_1$*, then* $1, \omega, \omega_1, \omega \times \omega_1$*, and* $[\omega_1]^{<\omega}$ *are the only cofinal types of directed sets of size at most* $\aleph_1$*.*

This next theorem of Todorcevic was originally presented in [4], and the proof was later simplified in [5]. Here we will take the latter approach, and along the way expand on the argument presented.

**Theorem 21** (Todorcevic)**.** *If CH holds, then there exists* $2^{\omega_1}$ *many distinct cofinal types of size* $\omega_1$*.*

Before proving this theorem, we will introduce some of the machinery that Todorcevic presented in his book to simplify this proof.
Let $X$ be a topological space. We will define the set

$$\mathcal{K}(X) = \{K \subseteq X : K \text{ is compact}\}.$$

We order $\mathcal{K}(X)$ by inclusion, and we note that $(\mathcal{K}(X), \subseteq)$ is a directed set.

Let us note that if $X$ is a compact space, then clearly $\mathcal{K}(X) =_T 1$. We will now study $\mathcal{K}(S)$ for $S$ a subspace of $\omega_1$, where $\omega_1$ is a topological space with its induced order topology. The practice of studying $\mathcal{K}(S)$ can be understood as the set-theoretic practice of "shooting a club through $S$". We also note that for $S \subset \omega_1$, then

$$\mathcal{K}(S) = \{C \in [S]^{\leq\omega} : C \text{ is closed}\}.$$

In particular, all previous cofinal types discussed thus far may be described as the set of compact subsets of some subspace of $\omega_1$. The following proposition states them explicitly.

**Proposition 22.** *Consider* $\omega_1$ *a topological space under the order topology. Then the following hold:*

1. $\mathcal{K}(\omega + 1) =_T 1$,
2. $\mathcal{K}(\omega) =_T \omega$,
3. $\mathcal{K}(\omega_1) =_T \omega_1$,
4. $\mathcal{K}(\omega_1 \setminus \{\omega\}) = \omega \times \omega_1$,
5. $\mathcal{K}(\{\alpha + 1 : \alpha < \omega_1\}) = [\omega_1]^{<\omega}$.

*Proof.* We see that $\omega+1$ is compact, so (*1*) holds. The set$\{n : n \in \omega\}$ is cofinal in $\mathcal{K}(\omega)$ and $\{\alpha+1 : \alpha < \omega_1\}$ is cofinal in $\mathcal{K}(\omega_1)$, so *2*) and (*3*) hold. Now, given $n \in \omega$ and $\alpha > \omega$, we construct

$$K_{n,\alpha} = \{0, \ldots, n\} \cup [\omega+1, \alpha].$$

We see that $\{K_{n,\alpha} : n \in \omega \text{ and } \alpha > \omega\}$, is a cofinal subset of $\mathcal{K}(\omega_1 \setminus \{\omega\})$ isomorphic to $\omega \times \omega_1$, so (*4*) holds. Lastly, the set $\{\{\alpha+1\} : \alpha < \omega_1\}$ has no bounded countable subsets, so (*5*) holds. ☐

Let NS be the ideal formed by non-stationary subsets of $\omega_1$. We will write $X \subseteq_{\mathrm{NS}} Y$ to mean that $X \setminus Y \in \mathrm{NS}$.

The following proposition makes use of sequences of elementary submodels. We will not elaborate on the study of the models here, and instead we direct the reader to [2].

**Proposition 23.** *Let $X, Y \subseteq \omega_1$ be bistationary. If $\mathcal{X} \leq_T \mathcal{Y}$, then $Y \subseteq_{\mathrm{NS}} X$.*

*Proof.* Suppose that $\mathcal{K}(X) \leq_T \mathcal{K}(Y)$ and that $Y \not\subseteq_{\mathrm{NS}} X$. That is, $Y \setminus X$ is nonstationary. Let $f : \mathcal{K}(X) \to \mathcal{K}(Y)$ be a Tukey map. Let $M$ be a countable elementary submodel of $H_{\omega_2}$ such that $X, Y, f \in M$ and that $\delta = M \cap \omega_1 \in Y \setminus X$. Consider an enumeration $\delta = \{\delta_n : n \in \omega\} \setminus \{0\}$ and let $\gamma \in X$ with $\gamma > \delta$. We note that $f(\{\gamma\}) \in \mathcal{K}(Y)$. In particular, $f(\{\gamma\})$ is closed. Now, we will construct recursively a sequence $\{\alpha_n : n \in \omega\} \subseteq M$ such that for ever $n \in \omega$:

1. If $n < m$ then $\alpha_n < \alpha_m$,
2. $\langle \alpha_m \rangle \to \delta$,
3. $\alpha_n \in Z$,
4. If $m < n$, then either $\delta_m \in f(\{\alpha_n\})$ or $\max(f(\{\alpha_n\}) \cap \delta_m) = c_m$.

We will let $\alpha_0 = \min(Y \cap M)$. We will use the elementarity of $M$ to construct the succesors $\alpha_{n+1} > \alpha_n$ satisfying the conditions above. We note that $f(\{\alpha_n\}) \in M$ for each $n \in \omega$. Hence, $f(\{\alpha_n\}) \subseteq \delta$. We may then define $K = \{\delta\} \cup \bigcup_{n\in\omega} f(\{\alpha_n\})$. We claim that $K \in \mathcal{K}(Y)$. Indeed, it is clear that $K \subseteq Y$. Let $\beta \notin K$, and we will show that $\beta$ is not an accumulation point of $K$. If it were, then this could only hold if $\beta < \delta$, so there must exists some $n \in \omega$ such that $\beta = \delta_n$. Therefore,

$$\bigcup K \cap \beta = \max \left\{ c_n, \left( \bigcup_{i<n} f(\{\alpha_i\}) \right) \cap \beta \right\} < \beta.$$

Since each $f(\{\alpha_i\})$ is compact, $\beta$ cannot be an accumulation point of them. Thus, $\{f(\{\alpha_n\}) : n \in \omega\} \subseteq \mathcal{K}(Y)$ is bounded, so their preimages must also be bounded in $\mathcal{K}(X)$. That is, there exists some $D \in \mathcal{K}(X)$. such that $\{\alpha_n : n \in \omega\} \subseteq D$. However, we then have that $\delta = \bigcup_{n\in\omega} \alpha_n \in D \subseteq X$, a contradiction. ☐

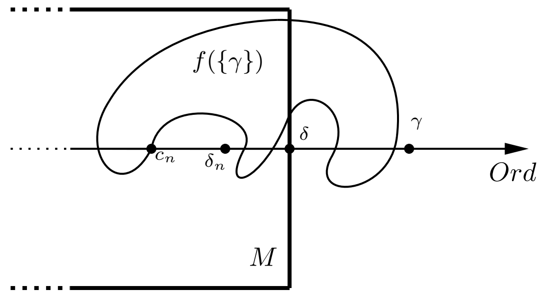


Figure 3: Visualizing the construction in Theorem 21

**Lemma 24.** *If $S$ is stationary, then $|\mathcal{K}(S)| = \mathfrak{c}$.*

**Theorem 25** (Ulam)**.** *There exists a disjoint family of sets $\{S_\alpha : \alpha < \omega_1\} \subseteq \mathrm{NS}^\dagger$.*

**Corollary 26.** *There exists a family $\{Z_\alpha : \alpha \in 2^{\omega_1} \subseteq \mathrm{NS}^\dagger$ such that if $\alpha \neq \beta$, then $Z_\alpha \neq_{\mathrm{NS}} Z_\beta$.*

**Corollary 27** (Todorcevic)**.** *There exists at least $2^{\omega_1}$ many cofinal types of size $\mathfrak{c}$.*

**Theorem 28** (Todorcevic)**.** *If* CH *holds, then there exists $2^{\omega_1}$ many cofinal types of size $\omega_1$.*

# 3 Bounding and Dominating Numbers

The study of the bounding and dominating numbers are closely related to the existence of cofinal types. Loosely speaking, results in combinatorial set theory surrounding the bounding and dominating numbers can tell us how many cofinal types may exist in certain models of set theory. This section will now build up the theory of bounding and dominating numbers.

Let $f, g \in \omega^\omega$, and let $m \in \omega$. We define the following relations on and notations for $\omega^\omega$:

1. $f \leq g$ if for all $n \in \omega$, $f(n) \leq g(n)$.
2. $f \leq^* g$ if for all but finitely many $n \in \omega$, $f(n) \leq g(n)$.
3. $f \leq_m g$ if for all $n \geq m$, $f(n) \leq g(n)$.
4. $f =^* g$ if for all but finitely many $n \in \omega$, $f(n) = g(n)$.
5. We denote a finite set of increasing functions on $\omega^\omega$ as $\omega^{<\omega\nearrow}$

Let $(P, \leq)$ be a partially ordered set. We recall that a set $B \subseteq P$ is unbounded if there does not exist a $p \in P$ such that $q \leq p$ for all $q \in B$. We say that a subset $D \subseteq P$ is *dominant* if for all $p \in P$, there exists a $q \in D$ such that $p \leq q$. Now, let $P$ be a partially ordered set without a maximal element. We define the *bounding* number of $P$ as

$$\mathfrak{b}(P) = \min\{\text{card}(B) : B \subseteq P \text{ is unbounded}\}.$$

Similarly, we define the *dominating number* of $P$ as

$$\mathfrak{d}(P) = \min\{\text{card}(D) : D \subseteq P \text{ is dominant}\}.$$

One may immediately note that $\mathfrak{b}(P) \leq \mathfrak{d}(P) \leq |P|$.

We now present some results regarding the bounding and dominating numbers, for $\omega^\omega$.

**Lemma 29.** $\mathfrak{b}(\omega^\omega, \leq) = \omega$.

*Proof.* Let $B = \{f_n : n \in \omega\}$ where each $f_n$ is the constant function at $n$. □

This shows us that the bounding number is rather trivial with the $\leq$ ordering over $\omega^\omega$. The situation becomes rather complicated when we consider the ordering $\leq^*$. We now define *the* bounding number as

$$\mathfrak{b} = \mathfrak{b}(\omega^\omega, \leq^*),$$

as well as *the* dominating number

$$\mathfrak{d} = \mathfrak{d}(\omega^\omega, \leq^*).$$

The properties of both the bounding and dominating number are rather mysterious.

**Proposition 30.** $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

*Proof.* It is clear that $\mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$. It is enough to see that if $B \subseteq \omega^\omega$ with $|B| = \omega$, then $B$ is bounded. Indeed, let $B = \{f_n : n \in \omega\}$, we define $g \in \omega^\omega$ by $g(n) = f_0(n) + \ldots + f_n(n)$. □

**Proposition 31.** $\mathfrak{d} = \mathfrak{d}(\omega^\omega, \leq)$.

*Proof.* It is clear that $\mathfrak{d} \leq \mathfrak{d}(\omega^\omega, \leq)$. Let $D \subseteq \omega^\omega$. If $D$ is $\leq^*$-dominant, then we define

$$\bar{D} = \{f \in \omega^\omega : \exists g \in D (f =^* g)\}.$$

We see that $\bar{D}$ is $\leq$-dominant, and that $|D| = |\bar{D}|$. □

**Proposition 32.** $\mathfrak{b}$ *is a regular cardinal.*

*Proof.* Suppose that $\kappa = \mathrm{cf}(\mathfrak{b}) < \mathfrak{b}$. Let $B \subseteq \omega^\omega$ be unbounded with size $\mathfrak{b}$. For all $\alpha \in \kappa$, we may find $B_\alpha$ such that $|B_\alpha| < \mathfrak{b}$ and $B = \bigcup_{\alpha<\kappa} B_\alpha$. Since $|B_\alpha| < \mathfrak{b}$, there exists an $f_\alpha$ that dominates $B_\alpha$. In particular, for all $g \in B_\alpha$, $g \leq f_\alpha$. As $\kappa < \mathfrak{b}$, there exists an $h \in \omega^\omega$ such that $f_\alpha \leq^* h$ for each $\alpha < \kappa$. It follows that $h$ dominates $B$, which is a contradiction. □

**Proposition 33.** $\mathfrak{b} \leq \mathrm{cf}(\mathfrak{d})$.

*Proof.* The proof proceeds like the one above. □

Statements about the arithmetic of bounding and dominating numbers are often independent of ZFC. We will now present some results surrounding this.

Let $\kappa$ be a cardinal and Let $D = \{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$. We say that $\mathfrak{b}$ is a *scale* if

1. Every $f_\alpha$ is an increasing function,
2. If $\alpha < \beta$, then $f_\alpha \leq^* f_\beta$,
3. $D$ is dominant.

**Proposition 34.** $\mathfrak{b} = \mathfrak{d}$ *if and only if there exists a scale.*

*Proof.* Let $D = \{d_\alpha : \alpha \in \mathfrak{d}\}$ be a family of dominant sets. We will recursively construct the set $B = \{f_\alpha : \alpha \in \mathfrak{d}\}$ such that

1. $f_\alpha$ is increasing,
2. If $\alpha < \beta$ then $f_\alpha \leq^* f_\beta$,
3. $d_\alpha \leq^* f_\alpha$.

Let $\alpha < \mathfrak{d} = \mathfrak{b}$. Then $\{f_\xi : \xi < \alpha\} \cup \{d_\alpha\}$ es bounded. We see that $f_\alpha$ dominates this family. Thus, $B$ is a scale.
Conversely, let $D$ be a scale of size $\kappa$. Let $B \subseteq \omega^\omega$ be an unbounded set of size $\mathfrak{b}$. Then, we may find $D' \subseteq D$ such that

1. $|D'| = \mathfrak{b}$,
2. For every $f \in B$ there exists a $g \in D'$ such that $f \leq^* g$.

Such a $D'$ exists since $D$ is dominant. Moreover, we claim that $D'$ is cofinal in $D$. Indeed, suppose it is not. Then there exists a $g \in D$ that is not dominated by any function in $D'$. However, as $D$ is well-ordered, $g$ must dominate $D$. Then, $g$ dominates $B$, a contradiction. Since $D'$ is cofinal, it is dominant. Therefore, $\mathfrak{d} \leq |D'| = \mathfrak{b}$. □

**Proposition 35.** *Let $\kappa$ be a regular cardinal. If $D = \{f_\alpha : \alpha < \kappa\}$ is a scale, then $\mathfrak{b} = \mathfrak{d} = \kappa$.*

*Proof.* Let $\kappa$ be a regular cardinal and $D$ a scale with $|D| = \kappa$. Let $D'$ be as constructed in the previous proposition. Since $\kappa$ is regular, we have that $\kappa = |D| = |D'| = \mathfrak{b} = \mathfrak{d}$. $\square$

The following result of Todorcevic in [3] shows that the bounding number is directly related to adding a new cofinal type.

**Theorem 36** (Todorcevic)**.** *If $\mathfrak{b} = \omega_1$, there is a sublattice $D_{\mathfrak{b}}$ of $\omega^\omega$ such that*

$$\omega \times \omega_1 <_T D_{\mathfrak{b}} <_T [\omega_1]^{<\omega}$$

*In particular, let $D_{\mathfrak{b}}$ be the sublattice of $\omega^\omega$ generated by an $<^*$-increasing $<^*$-unbounded sequence $\{f_\xi : \xi < \mathfrak{b}\} \subseteq \omega^{\uparrow\omega}$*

# 4 Classifying Cofinal Types in Models of ZFC

## 4.1 Introduction to Cohen Forcing

We will now discuss the Cohen forcing notion, described with a treatment á la Jech in [1]. This section will simply build up the theory of cohen forcing, following the development of the topic in [1]. This forcing notion was used to adjoin $\omega_2$ many real numbers to a ground model. In general, Cohen forcing will give us the procedure to adjoin $\kappa$ many real numbers, known as *Cohen reals*, to a ground model.

let $M$ be a countable transitive model, and let $\kappa$ be an infinite cardinal in $M$. Let $\mathbb{P}$ be the set of functions such that

1. $\mathrm{dom}(p)$ is a finite subset of $\kappa \times \omega$,
2. $\mathrm{ran}(p) \subset \{0, 1\}$,

and let $p$ be stronger than $q$ if and only if $q \subset p$. Now, let $G$ be a generic set of conditions on $\mathbb{P}$, and let $f = \bigcup G$. Since $G$ is a filter, it follows that $f$ is a function. It may be shown by a genericity argument that $f$ is a function from $\kappa \times \omega$ into $\{0, 1\}$. For each $\alpha < \kappa$, let $f_\alpha$ be the function on $\omega$ defined by $f_\alpha(n) = f(\alpha, n)$, and let $a_\alpha = \{n \in \omega : f_\alpha(n) = 1\}$. Each $a_\alpha$ is a subset of $\omega$, hence some characterization of a real number.

**Proposition 37.** *For all $\alpha < \kappa, f_\alpha \notin M$.*

*Proof.* Suppose that $\alpha < \kappa$ and that $f_\alpha \in M$. Let $D = \{p \in \mathbb{P} : \exists n \in \omega[(\alpha, n) \in \mathrm{dom}(p) \wedge p(\alpha, n) \neq f_\alpha(n)]\}$. We see that $D$ is dense in $\mathbb{P}$, hence $f(\alpha, n) \neq f_\alpha(n)$, a contradiction. $\square$

**Proposition 38.** $\left(2^{\aleph_0}\right)^{M[G]} \geq \left(\kappa^{\aleph_0}\right)^M$.

*Proof.*

$$\left(2^{\aleph_0}\right)^{M[G]} = \left(\left(2^{\aleph_0}\right)\right)^{M[G]} \geq \left(\kappa^{\aleph_0}\right)^{M[G]} \geq \left(\kappa^{\aleph_0}\right)^M.$$

$\square$

The Cohen forcing notion actually forces the continuum to be exactly $\kappa^{\aleph_0}$.

**Lemma 39.** *Let $\lambda$ be a cardinal in $M$ and $G$ a generic ultrafilter on a complete Boolean algebra $B$. Then $\left(2^\lambda\right)^{M[G]} \leq \left(|B|^\lambda\right)^M$.*

*Proof.* Working in $M$, let $X$ be the set of functions $f : \lambda \to B$ such that there exists $\dot{A} \in M^B$ such for $f(\alpha) = \|\check{\alpha} \in \dot{A}\|$ for all $\alpha < \lambda$. In $M[G]$, for each $f \in X$ we choose an $\dot{A}$ satisfying this, and define $g(f) = \dot{A}_G$. In $M[G]$, we have that $\mathcal{P}(\lambda) \subseteq \operatorname{ran}(g)$. If $A \subseteq \lambda$, we choose $\dot{A}$ such that $\dot{A}_G = A$. We then define $f(\alpha) = \|\check{\alpha} \in \dot{A}\|$ for all $\alpha < \lambda$. Therefore, $g(f) = A$ and thus $\left(2^\lambda\right)^{M[G]} \leq |X|^M \leq \left(|B|^\lambda\right)^M$. □

**Theorem 40.** *Let $\kappa$ be an infinite cardinal, and let $\mathbb{P}$ be the forcing notion above. Let $G$ be $\mathbb{P}$-generic over $M$. Then, $\left(2^{\aleph_0}\right)^{M[G]} = \left(\kappa^{\aleph_0}\right)^M$.*

*Proof.* Let $B$ be the complete Boolean algebra of regular open subsets of $\mathbb{P}$ with respect to the order topology. Since $\mathbb{P}$ satisfies the countable chain condition, $|B| = \kappa^{\aleph_0}$. Therefore, $\left(2^{\aleph_0}\right)^{M[G]} \geq \left(\kappa^{\aleph_0}\right)^M$. The other direction follows from above. □

It remains to show that $\mathbb{P}$ as a forcing notion preserves cardinals. In particular, $\mathbb{P}$ satisfies the countable-chain condition, and so $\mathbb{P}$ preserves both cardinals and cofinalities. The following result tell us when cardinals are preserved in generic extensions.

**Lemma 41.** *Suppose that $\alpha$ is a limit ordinal, and $\kappa$and $\lambda$ are regular cardinals. Let $f : \kappa \to \alpha$ be a strictly increasing function with $\operatorname{ran}(f)$ cofinal in $\alpha$, and $g : \lambda \to \alpha$ strictly increasing with $\operatorname{ran}(g)$ cofinal in $\alpha$. Then $\kappa = \lambda$.*

*Proof.* Suppose not. Without loss of generality, suppose that $\kappa < \lambda$. For each $\xi < \kappa$, let $\eta_\xi < \lambda$ such that $f(\xi) < g(\eta_\xi)$. We define $\rho = \sup_{\xi<\kappa} \eta_\xi$. Then $\rho < \lambda$ since $\lambda$ is regular. However, $f(\xi) < g(\rho) < \alpha$ for all $\xi < \kappa$, a contradiction. □

**Theorem 42.** *Let $M$ be a countable transitive model of ZFC. Let $\mathbb{P}$ be a notion of forcing in $M$, and let $\kappa$ be a cardinal in $M$.*

1. *If $\mathbb{P}$ preserves regular cardinals $\geq \kappa$, then it preserves cofinalities $\geq \kappa$.*
2. *If $\mathbb{P}$ preserves cofinalities $\geq \kappa$ and $\kappa$ is regular, then $\mathbb{P}$ preserves cardinals $\geq \kappa$.*
3. *If $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.*

*Proof.* Firstly, suppose that $\mathbb{P}$ preserves regular cardinals $\geq \kappa$. Let $\alpha \in M$ be a limit ordinal with $(\operatorname{cf}(\alpha))^M \geq \kappa$. Then $(\operatorname{cf}(\alpha))^M$ is a regular in $M$ which is $\geq \kappa$ and hence also regular in $M[G]$. By the lemma above, we have that $(\operatorname{cf}(\alpha))^M = (\operatorname{cf}(\alpha))^{M[G]}$.
Secondly, suppose that $\mathbb{P}$ preserves cofinalities $\geq \kappa$ and that $\kappa$ is regular, but suppose that cardinals $\geq \kappa$ are not preserved. Then, let $\lambda$ be the least cardinal

$\geq \kappa$ in $M$ that is not a cardinal in $M[G]$. If $\lambda$ is regular in $M$, then $\lambda = (\text{cf}(\lambda))^M = (\text{cf}(\lambda))^{M[G]}$, implying that $\lambda$ is regular in $M[G]$, a contradiction. Instead if $\lambda$ is singular in $M$, then $\lambda > \kappa$ as $\kappa$ is regular and $\lambda \geq \kappa$. That is, $\lambda$ is the union of some set $X$ of cardinals of $M$ which are regular $\geq \kappa$, so the elements of $X$ are cardinals in $M[G]$. However, $\lambda$ is chosen to be minimal, implying that $\lambda$ is a cardinal in $M[G]$, a contradiction. Lastly, suppose that $\mathbb{P}$ preserves cofinalities, then the above argument suffices when $\kappa = \omega$. □

### 4.2 Forcing a Directed Set on $\omega_1$

The notion of Cohen forcing may be used to create a generic extension with a new directed partial order on $\omega_1$. We describe the forcing notion here.

Let $\mathbb{P}$ be the set of ordered pairs $p = (P, \leq_p)$ such that

1. $P$ is a finite subset of $\omega_1$.
2. $\leq_p$ is a partial order on $P$. In particular, if $a, b \in P$ with $a \leq_p b$, then $a < b$ as ordinals.

We partially order $\mathbb{P}$ with $<$, where $(P, \leq_p) < (Q, \leq_q)$ if and only if $Q \subset P$ and $\leq_p$ restricted to $Q$ is $\leq_q$.

**Theorem 43.** *Let $M$ be a ground model of ZFC, and $G$ a $\mathbb{P}$-generic filter in $M$ with respect to the forcing notion above. Then there exists a partial order $\leq_D$ on $\omega_1$ in $M[G]$ with $\leq_D \notin M$.*

*Proof.* Let $G$ be the generic filter in question. Then $\bigcup G = D \times \leq_D$, where $D \subseteq \omega_1$ and $\leq_D \subseteq \omega_1 \times \omega_1$. We see that $D = \omega$. Indeed, let $\alpha < \omega$, and let $E = \{p \in \mathbb{P} : p \in \text{dom}(p)\}$. Since $E$ is dense and therefore meets $G$, we have that $\alpha \in D$. We also see that $\leq_D$ is a partial order on $\omega_1$. Indeed, $\leq_D$ is reflexive by the same density argument. Let $a, b \in \omega_1$ such that $a \leq_D b$ and $b \leq_D a$. Then there exists conditions $p, q \in G$ such that $a, b \in \text{dom}(p)$ and $a, b \in \text{dom}(q)$ such that $a \leq_p b$ and $b \leq_p a$. Since $G$ is a filter, there exists a condition $r \in G$ such that $r \leq p$ and $r \leq q$. That is $a, b \in \text{dom}(r)$ and $a \leq_r b$ and $b \leq_r a$. However, $\leq_r$ is a partial order by definition, so we have that $a = b$. Since $r \in G$, we have that $\leq_D$ is antisymmetric. Lastly, let $a \leq_D b \leq_D c$ where $a, b, c \in \omega_1$. Then there exists condition $p, q \in \mathbb{P}$ such that $a, b \in \text{dom}(p)$ with $a \leq_p b$ and $b, c \in \text{dom}(q)$ with $b \leq_p c$. Since $G$ is a filter, there exists an $r \in G$ extending $p$ and $q$. That is, $a, b, c \in \text{dom}(r)$ and $(a, b), (b, c) \in \leq_r$. However, $\leq_r$ is a partial order on $\text{dom}(p)$, so $(a, c) \in \leq_r$. Since $r \in G$, we have that $\leq_D$ is transitive. Thus, $(\omega_1, \leq_D)$ is a directed set. □

Let $M$ be a ground model of ZFC + (G)CH. Let $\mathbb{P}$ be the forcing notion adjoining $\aleph_2$ many Cohen reals to $M$. Let $G$ be a $\mathbb{P}$-generic filter on $M$. We call $M[G]$ the Cohen model. It is currently unknown as to how many cofinal types exist in the Cohen model. The follows result of Todorcevic in [3] tells us that there are at least six distinct types of size at most $\omega_1$.

**Theorem 44.** *Let $M[G]$ be the Cohen model built on some ground model of* ZFC + (G)CH. *Then* $\mathfrak{b}^{M[G]} = \omega_1$. □

**Corollary 45.** *There are at least six distinct cofinal types of size at most $\omega_1$ in the Cohen model.*

### 4.3 Future Work

There are still several remaining questions regarding this new directed set and the Cohen model. In particular, it is entirely possible that the forcing that adjoins a directed partial order on $\omega_1$ is not cofinally equivalent to any cofinal type in the ground model. Moreover, in the Cohen model, it is not known how the directed sets $\mathcal{K}(X)$ introduced by Todorcevic behave cofinally with respect to Cohen forcing. We will end this paper with the following conjectures.

**Conjecture 46.** *Let $M$ be a ground model of* ZFC + CH. *Let $\mathbb{P}$ be the forcing notion that adjoins a directed partial order $\leq_G$ on $\omega_1$. Let $D$ be a directed set of size at most $\omega_1$ in $M$. Then $D \neq_T (\omega_1, \leq_G)$.*

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