Composition Operators on a Hilbert Subspace of a Banach Algebra

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COMPOSITION OPERATORS ON A HILBERT SUBSPACE OF A BANACH ALGEBRA

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Composition Operators on a Hilbert Subspace of a Banach Algebra

by

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A thesis submitted in partial fulfillment for the
degree of Bachelor of Arts

in the
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TRINITY UNIVERSITY

Abstract

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Our investigation is two-fold. On one hand, we aim for a self-contained introduction to the theory necessary in understanding general measure spaces, Banach and Hilbert spaces, and C^*-algebras. On the other hand, inspired by the study of a topological space of composition operators on a weighted Banach algebra of bounded functions on an unbounded, locally finite metric space, we construct a separable Hilbert subspace of this algebra and consider composition operators on this space. Firstly, establish conditions for such operators to be elements of the C^*-algebra of bounded, linear operators on this Hilbert space. Secondly, we take the weight to be the counting measure and identify our underlying metric space with $\mathbb{N}$ and establish results concerning the adjoint, invertibility, and the unitary composition operators. Finally, we analyze the spectrum of these operators. We establish two results that provide an avenue of investigation for the spectral structure of this class of operators that will hopefully be fruitful in further work.

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Contents

Dedicated to God, my Mother, and my Father. For, without you, none of this would be possible.

Chapter 1

Measure Spaces

We begin by abstracting the most fundamental properties of Lebesgue measure on $\mathbb{R}$ in the absence of any imposition of a topology. Consequentially, the theory highlighted in this chapter will remain true in every system where the given axioms are satisfied. Recall that to establish countable additivity on $\mathbb{R}$ of the Lebesgue measure defined on a σ-algebra, you begin with fundamental concepts from set-theory. Defining a set function assigning length to every bounded interval in $\mathbb{R}$, then extending to the outer measure defined on the Borel algebra, and then restricted this outer measure to the σ-algebra of measurable sets to obtain a new measure. This is based on the Carathéodory construction of Lebesgue measure. Inspired by this, we highlight this construction for a general abstract set X and elucidate the most important properties pertaining to our investigation.

1.1 Measures

Definition 1.1. A *measurable* space is a couple $(X, \mathcal{M})$ consisting of a set X and a σ-algebra $\mathcal{M}$ of subsets of X. A subset $E \subseteq X$ is measurable if $E \in \mathcal{M}$.

Definition 1.2. A *measure* $\mu : \mathcal{M} \to [0, \infty]$ on a measure space $(X, \mathcal{M})$ is a set function for which $\mu(\emptyset) = 0$ and that is countably additive in the sense that for any countable collection $\{E_\lambda\}_{\lambda\in\Lambda}$ of disjoint, measurable sets,

$$\mu\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right) = \sum_{\lambda\in\Lambda} \mu(E_\lambda).$$

A measurable space $(X, \mathcal{M})$ together with a measure μ is a *measure* space, $(X, \mathcal{M}, \mu)$.

Example 1.1. Recall that a subset $E \subseteq \mathbb{R}^n$ is Lebesgue measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

for all $A \subseteq \mathbb{R}^n$, where μ^* is the outer measure. Denote this collection by $\mathcal{L}(\mathbb{R}^n)$. This set is a σ-algebra, and if we define $\mu := \mu^*|_{\mathcal{L}(\mathbb{R}^n)}$, then μ is the Lebesgue measure. The triple $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \mu)$ is then a measure space.

Example 1.2. Let X be any non-empty set and consider $\mathscr{P}(X)$, its power set. Then, $(X, \mathscr{P}(X))$ is a measurable space. Let $x \in X$. Then, the Dirac measure concentrated at x, δ_x, defines the Dirac measure space $(X, \mathscr{P}(X), \delta_x)$. Specifically, this is a probability space.

Proposition 1.3. Let $(X, \mathcal{M}, \mu)$ be any measure space.

(Finite Additivity) For any finite, disjoint collection $\{E_k\}_{k=1}^n$ of measurable sets,

$$\mu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu(E_k).$$

(Monotonicity) IF $A, B \in \mathcal{M}$, and $A \subseteq B$, then

$$\mu(A) \leq \mu(B).$$

(Excision) If, moreover, $A \subseteq B$ and A is of finite measure, then

$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

If A is of measure zero, then $\mu(B \setminus A) = \mu(B)$.

(Countable Monotonicity) For any countable collection $\{E_\lambda\}_{\lambda \in \Lambda}$ such that

$$E \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda,$$

where E is measurable, then

$$\mu(E) \leq \sum_{\lambda \in \Lambda} \mu(E_\lambda).$$

Observe that the countable monotonicity is an amalgamation of the properties of countable additivity and monotonicity. Moreover, given a measure space $(X, \mathcal{M}, \mu)$, the measure μ is continuous in the sense that if $\{A_k\}$ is an ascending sequence of measurable sets, then

$$\mu\left(\bigcup_{k=1}^\infty A_k\right) = \lim_{k \to \infty} \mu(A_k).$$

If $\{B_k\}$ is a descending sequence of measurable sets for which B_1 is of finite measure, then

$$\mu\left(\bigcap_{k=1}^\infty B_k\right) = \lim_{k \to \infty} \mu(B_k).$$

Definition 1.4. For a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, we say that a property holds *almost everywhere* on E provided it holds on $E \setminus E_0$, where $E_0 \in \mathcal{M}$ is of measure zero.

Definition 1.5. Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is called *finite* provided that $\mu(X) < \infty$. It is called σ-finite provided

$$X = \bigcup_{\omega \in \Omega} E_\omega,$$

where $\{E_\omega\}_{\omega \in \Omega}$ is a countable collection of measurable sets, with $\mu(E_\omega) < \infty$ for all $\omega \in \Omega$.

1.1.1 Outer Measures and Carathéodory Extension

Recall that in the construction of Lebesgue outer measure on subsets of $\mathbb{R}$, you firstly define a primitive set function assigning length to every bounded interval. Subsequently, you define the outer measure of a set taking the infimum of all sums of lengths of countable collections of bounded intervals that cover the set.

Theorem 1.6. Let $\mathcal{S}$ be a collection of subsets of a non-empty set X and define $\mu : \mathcal{S} \to [0, \infty]$. Moreover, define $\mu^*(\emptyset) = 0$ and for all non-empty $E \subseteq X$, set

$$\mu^*(E) := \inf \sum_{k=1}^{\infty} \mu(E_k),$$

where the infimum is taken over all countable collections of sets in $\mathcal{S}$ covering E. Then, the set function $\mu^* : S^X \to [0, \infty]$ is an outer measure called the outer measure induced by μ.

Definition 1.7. Let $\mathcal{S}$ be a collection of subsets of X, μ a non-negative, extended real-valued set function defined on $\mathcal{S}$, and μ^* the outer measure induced by μ. The measure $\lambda = \mu^*|_{\mathcal{M}}$, where $\mathcal{M}$ is the σ-algebra of μ^* measurable sets, is the *Carathéodory measure induced by* μ.

Now, if we take a non-empty collection $\mathcal{S}$ of subsets of a set X and consider a set function $\mu : \mathcal{S} \to [0, \infty]$, it is natural to ask what properties must the aforementioned collection and the function have in order that the Carathéodory measure induced by μ be an extension of μ. Recall that a set function $\mu : \mathcal{S} \to [0, \infty]$ is said to be *finitely additive* if whenever $\{E_k\}_{k=1}^n$ is a finite, disjoint collection of sets in $\mathcal{S}$ whose union is also in $\mathcal{S}$, then

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k).$$

Definition 1.8. Let $\mathcal{S}$ be a collection of subsets of X and μ a set function. Then, μ is called a *premeasure* provided that it both finitely additive and countably monotone, and if $\emptyset \in \mathcal{S}$, then $\mu(\emptyset) = 0$.

Theorem 1.9. Let $\mathcal{S}$ be a non-empty collection of subsets of a set X that is closed with respect to the formation of relative complements and let $\mu : \mathcal{S} \to [0, \infty]$ be a premeasure. Then, the Carathéodory measure $\lambda : \mathcal{M} \to [0, \infty]$ induced by μ is an extension of μ.

It turns out that a number of premeasures are defined on collections of sets that are not closed under formation of relative complements. Consider the premeasure length defined on the collection of all bounded intervals in $\mathbb{R}$. Because of this, the notion of a semiring becomes fruitful as it provides the property that every premeasure defined on it has a unique extension to a premeasure on a collection of sets which is closed with respect to the formation of relative complements. This will pave the way for an introduction to algebras to be had later, which will be of chief interest in the latter portion of our two-fold investigation.

Definition 1.10. A non-empty collection $\mathcal{S}$ of subsets of a set X is called a *semiring* if whenever $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$ and there exists a finite, disjoint collection $\{C_k\}_{k=1}^n \subseteq \mathcal{S}$ such that

$$A \setminus B = \bigcup_{k=1}^{n} C_k.$$

Theorem 1.11. (The Carathéodory-Hahn Theorem). Let $\mathcal{S}$ be a semiring of subsets of X and define a premeasure $\mu : \mathcal{S} \to [0, \infty]$. Then, the Carathéodory measure λ induced by μ is an extension of μ. Moreover, if μ is σ-finite, then so is λ, and λ is the unique measure on the σ-algebra of μ^*-measurable sets extending μ.

1.2 Integration

Now that we have established the above results regarding measures and their extensions, we can begin in understanding the action of integrating over a general measure space. In considering measurable functions, the investigation of integration is similar to the development considered in the study of Lebesgue measurable functions of a single, real variable. Recall that if given a measurable space $(X, \mathcal{M})$, and a real-valued function f on X, then f is said to be measurable if, and only if, the preimage of every open set is measurable. Here, we will begin with the investigation of measurable functions to motivate the general theory of integration on abstract measure spaces. Firstly, consider an elucidating example about measurable functions.

Example 1.3. Let X be a set and take for a σ-algebra $\mathcal{M} = 2^X$— the set of all subsets of X. Then, every extended real-valued function on X is measurable with respect to $\mathcal{M}$. If, instead, we consider the smallest σ-algebra $\mathcal{M} = \{X, \emptyset\}$, then the only measurable functions are the constant maps. Considering (X, τ) as a topogical space, and if $\mathcal{M}$ is the σ-algebra of subsets of X such that $\tau \subseteq \mathcal{M}$, then every continuous real-valued function on X is measurable with respect to $\mathcal{M}$.

Definition 1.12. Let $(X, \mathcal{M})$ be a measurable space. A function $\psi : X \to \mathbb{R}$ is *simple* if there exsits a finite collection $\{E_k\} \subseteq \mathcal{M}$ of measurable sets and a corresponding set $\{a_k\} \subseteq \mathbb{R}$ for which

$$\psi = \sum_{k=1}^{n} a_k \chi_{E_k}.$$

Notice that a simple function is measurable and takes a finite number of values.

Lemma 1.13. Let $(X, \mathcal{M})$ be a measurable space and f a measurable, bounded function on X. Then, for every $\epsilon > 0$, there exists simple functions ψ_ϵ and φ_ϵ defined on X for which

$$\varphi_\epsilon \leq f \leq \psi_\epsilon \quad \text{and} \quad 0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon,$$

on X.

Theorem 1.14. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on X. Then, there exists a sequence $\{\psi_n\}$ of simple function on X converging pointwise on X to f and is such that $|\psi_n| \leq |f|$ on X for all n.

In the theorem above, if X is σ-finite, we may select the sequence $\{\psi_n\}$ so that every ψ_n vanishes outside a set of finite measure. Moreover, if f is non-negative on X, we may select the sequence $\{\psi_n\}$ to be increasing and every $\psi_n \geq 0$ on X.

Theorem 1.15. (Egoroff). Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence of measurable functions defined on X converging pointwise almost everywhere on X to a function f that is finite almost everywhere. Then, for every $\epsilon > 0$, there exists a measurable $X_\epsilon \subseteq X$ such that $f_n \to f$ uniformly on X_ϵ and $\mu(X \setminus X_\epsilon) < \epsilon$.

In the development of integration of a Lebesgue measurable function of a real variable with respect to Lebesgue measure, you firstly define the integral of a simple function over a set of finite Lebesgue measure. Secondly, you construct integrability of a bounded function on a set of finite measure and use an approximation lemma to show that a bounded, measurable function that vanishes outside a set of finite Lebesgue measure is integrable. Thirdly, you define the Lebesgue integral of a non-negative Lebesgue measurable function over an arbitrary Lebesgue measurable set to be the supremum of the integral of a dominated function over the set which vanished outside a set of finite Lebesgue measure. This construction is quite clear, and it would seem natural to try to extend to our case. However, this approach will not be fruitful for us. The reason?

Consider a measure space $(X, \mathcal{M}, \mu)$. If $\mu(X) = \infty$, we certainly want the integral of the constant one function over X to be infinite. However, if X is non-empty, and if we take the trivial σ-algebra $\mathcal{M} = \{X, \emptyset\}$, and we define $\mu(\emptyset) = 0$ and $\mu(X) = \infty$, then the only measurable function g vanishing outside a set of finite measure is $g \equiv 0$. Hence, the supremum of $\int_X g \, \mathrm{d}\mu$ ranging over all such functions is zero. Something else is needed in our consideration. To circumvent this, for our purposes, we will define the integral of non-negative simple functions and subsequently pivot to defining the integral of non-negative measurable functions in terms of integrals of non-negative simple functions.

Definition 1.16. Let $(X, \mathcal{M}, \mu)$ be a measure space and ψ a non-negative simple function on X. Define the integral of ψ over X as follows: if $\psi \equiv 0$ on X, define

$$\int_X \psi \, \mathrm{d}\mu = 0.$$

Otherwise, let $\{a_1, a_2, \ldots, a_n\}$ be positive values taken by ψ on X and, for every $1 \leq k \leq n$, let

$$E_k = \{x \in X | \psi(x) = a_k\}.$$

Define

$$\int_X \psi \, \mathrm{d}\mu = \sum_{k=1}^{n} a_k \mu(E_k).$$

For $E \subseteq X$ measurable, the integral of ψ over E with respect to μ is

$$\int_E \psi \, \mathrm{d}\mu = \int_X \psi \cdot \chi_E \, \mathrm{d}\mu.$$

Definition 1.17. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to [0, \infty]$ measurable. The integral of f over X with respect to μ is defined as

$$\int_X f \, \mathrm{d}\mu = \sup \left\{ \int_X \varphi \, \mathrm{d}\mu \;\middle|\; \varphi \text{ simple and } 0 \leq \varphi \leq f \text{ on } X \right\}$$

Definition 1.18. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a non-negative, measurable function on X. Then, f is *integrable* over X with respect to μ provided that

$$\int_X f \, \mathrm{d}\mu < \infty.$$

Now, we can turn to integration of any measurable function. Let $(X, \mathcal{M})$ be a measurable space and f a measurable function on X. Define the positive and negative part of f by f^+ and f^-, respectively, where

$$f^+ := \max\{f, 0\} \quad \text{and} \quad f^- := \max\{-f, 0\}$$

on X. Then, given a measure space $(X, \mathcal{M}, \mu)$, a measurable function f on X is integrable over X with respect to μ provided $|f| = f^+ + f^-$ is integrable over X with respect to μ. For such a function, by linearity, we define its integral as a difference of the integrals of its positive and negative parts, respectively.

Theorem 1.19. Let $(X, \mathcal{M}, \mu)$ be a measure space, f integrable over X, and $\{X_n\}$ a disjoint, countable collection of measurable sets for which

$$X = \bigcup_{n=1}^{\infty} X_n.$$

Then,

$$\int_X f \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_{X_n} f \, \mathrm{d}\mu.$$

Notice that we have only considered a class of integrable, simple functions vanishing outside a set of finite measure. Therefore, we give the following theorem.

Theorem 1.20. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function on X. If f is bounded on X and vanishes outside a set of finite measure, then f is integrable over X.

Theorem 1.21. (Lebesgue Dominated Convergence Theorem). Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions on X for which $f_n \to f$ pointwise almost everywhere on X, and f is measurable. Assume there exists a non-negative function g, integrable over X, and dominating $\{f_n\}$ on X. Then, f is integrable over X and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Corollary 1.22. Let X be a compact topological space and $\mathcal{M}$ a σ-algebra of subsets of X containing the topology endowed on X. If $f : X \to \mathbb{R}$ is continuous, and $(X, \mathcal{M}, \mu)$ is a finite measure space, then f is integrable over X with respect to μ.

1.3 Radon-Nikodym

Having established a sense for integration over general measure spaces, and the robustness of the Lebesgue integral, we give insight into the Radon-Nikodym theorem and its details that will be useful in the latter sections of this manuscript. To that effect, let $(X, \mathcal{M})$ be a measurable space. Given a measure μ on $(X, \mathcal{M})$ and a function $f : X \to [0, \infty]$, measurable with respect to $\mathcal{M}$, define the set function ν on $\mathcal{M}$ by

$$\nu(E) = \int_E f \, d\mu,$$

for all measurable sets $E \in \mathcal{M}$. The fact that ν is a measure on $(X, \mathcal{M})$ follows from linearity of integration and an application of the Monotone Convergence Theorem.

Definition 1.23. Let μ and ν be two measures defined on the same measurable space $(X, \mathcal{M})$. Then, ν is said to be *absolutely continuous* with respect to μ provided that for all $E \in \mathcal{M}$, if $\mu(E) = 0$, then $\nu(E) = 0$. We denote this by $\nu \ll \mu$.

Proposition 1.24. Let $(X, \mathcal{M}, \mu)$ be a measure space and ν a finite measure on the measurable space $(X, \mathcal{M})$. Then, ν is absolutely continuous with respect to μ if, and only if, for every $\epsilon > 0$, there is a corresponding $\delta > 0$ such that for any measurable $E \in \mathcal{M}$, if $\mu(E) < \delta$, then $\nu(E) < \epsilon$.

Theorem 1.25. (The Radon-Nikodym Theorem.) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $\nu \ll \mu$ a σ-finite measure defined on the measurable space $(X, \mathcal{M})$. Then, there exists a non-negative function f on X which is measurable with respect to $\mathcal{M}$ and for which

$$\nu(E) = \int_E f \, d\mu,$$

for all $E \in \mathcal{M}$.

Before moving on, we will mention the relationship between absolutel continuity of one measure with respect to another and their integral representations and the indefinite integral representation of an absolutely continuous function in terms of its derivative. We will shine light on the Radon-Nikodym derivative through an example that is central to Probability theory.

Example 1.4. Recall that a function F is an indefinite integral if, and only if, it is absolutely continuous. Its representation is then

$$F(x) = \int_a^x F'(t) \, dt + F(a).$$

Let $A = [a, b]$ be a closed, bounded interval and consider $\psi : A \to \mathbb{R}$ to be absolutely continuous. Then,

$$\psi(d) - \psi(c) = \int_c^d \psi' \, \mathrm{d}\mu,$$

for all $[c, d] \subseteq A$. Observe that this is sufficient to establish the Raodn-Nikodym Theorem if we consider $\mathcal{M}$ as the σ-algebra of Borel subsets of A and take μ as the Lebesgue measure on $\mathcal{M}$. Notice that if we take a finite measure ν on the measurable space $(A, \mathcal{M})$ which is absolutely continuous with respect to μ, and define a function ϕ on A by

$$\phi(x) = \nu([a, x]),$$

for all $x \in A$, then ϕ inherits absolute continuity from ν. We call ϕ the *cummulative distribution function associated to* ν. Hence, for all $E = [c, d] \subseteq A$, we have that

$$\nu(E) = \int_E \phi' \, \mathrm{d}\mu.$$

Because two σ-finite measures agreeing on compact subintervals of A agree on the smallest σ-algebra containing such intervals, then the above representation is true for all measurable sets E. Thus, the probability density function of a random variable is simply the Radon-Nikodym derivative of the induced measure with respect to a base measure—typically, Lebesgue measure.

Chapter 2

Banach Spaces

Let us detail some history. Banach spaces, introduced and investigated by Polish mathematician Stefan Banach, play a central role in functional analysis. These spaces arose from the study of function spaces by German mathematician David Hilbert, French mathematician Maurice René Fréchet, and the influential Hungarian mathematician Frigyes Riesz. These spaces are very rich, and a notable example to keep in mind is that of $\mathbb{R}^n$ with the usual Euclidean topology. Here, we will define what a Banach space is, consider some examples, state notable results, and then define the L^p spaces and their theory to illuminate their necessity in our analysis to come.

Definition 2.1. A *linear space* X over a field K is one which $(X, +)$ forms an Abelian group, has a multiplicative identity, vector multiplication is distributive over scalar addition, scalar multiplication is distributive over vector addition, and scalar multiplication is associative with respect to scalar multiples of vectors.

Definition 2.2. A *norm* on a linear space X over a field K is a function $||\cdot|| : X \to \mathbb{R}$ satisfying:

1. $||x|| \geq 0$ for all $x \in X$, and $||x|| = 0 \iff x = 0$.
2. $||\lambda x|| = |\lambda| \cdot ||x||$, for all $\lambda \in K$ and all $x \in X$.
3. $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in X$.

We call the tuple $(X, ||\cdot||)$ a *normed linear space.*

Given any normed linear space, a convenient metric is always available. For any two vectors x and y in the space, one can take as the distance between them

$$d(x, y) = ||x - y||.$$

Recall that if in a metric space (X, d), the sequence $\{x_n\}$ has the property that, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N \in \mathbb{N}$, then $d(x_n, x_m) < \epsilon$, the sequence is said to be a *Cauchy* sequence. In the case of $\mathbb{R}$, every Cauchy sequence converges. If a metric space

(X, d) does have the property that every Cauchy sequence in X converges in X, then the space is called a *Complete* metric space.

Moreover, with the above metric in mind, we can speak of normed linear spaces as metric spaces. The illustrated metric has the properties of translation invariance and homogeneity.

Definition 2.3. Let $(X, ||\cdot||)$ be a normed linear space. If (X, d) is complete under the norm-induced metric, then X is said to be a *Banach Space.*

Example 2.1. The space $C(X)$ of all continuous, real-valued (or complex-valued) functions defined on a compact metric space X under the sup-norm is a Banach Space.

Example 2.2. For $1 \leq p < \infty$, the sequence space $\ell^p(\mathbb{N})$ is the collection of all sequences x that are absolutely p-summable. That is, sequences for which

$$\sum_{i=1}^{\infty} |x_i|^p < \infty.$$

When endowed with the norm

$$||x||_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p},$$

then $\ell^p(\mathbb{N})$ becomes a Banach space.

The above example will be the basis of our analysis in chapter 5, so the reader is advised to keep this example in their back pocket for the remainder of the manuscript. Let us now turn to L^p spaces.

2.1 L^p Spaces

Function spaces are central in many questions in analysis. Of these, L^p spaces are of special importance to us, and especially when $p = 2$. Here, we will provide some of the fundamental background necessary for the material to come later.

To that effect, let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. If $1 \leq p \leq \infty$, define $L^p(X, \mathcal{M}, \mu)$ as the space of all complex-valued, measurable functions on X such that

$$\int_X |f|^p \, \mathrm{d}\mu < \infty.$$

For any $f \in L^p(X, \mathcal{M}, \mu)$, define the L^p norm of f by

$$||f||_p = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{1/p}.$$

Notice that $L^p(X, \mathcal{M}, \mu)$ is complete in the norm $||\cdot||_p$, and is thus a Banach space. Moreover, when $p = 2$, we have a Hilbert space— this will be of chief interest for us. Furthermore, there is a technicality we must discuss. The issue is that $||f||_p = 0$ does not imply that $f \equiv 0$ on X, but only that $f \equiv 0$ almost everywhere $[\mu]$. Hence, the definition of the above L^p space requires us to introduce an equivalence relation in which two functions are equivalent if they are the same μ-almost everywhere. We now introduce important results.

Theorem 2.4 (**Hölder**)**.** Suppose $1 < p < \infty$ and $1 < q < \infty$ are conjugate exponents. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and

$$||fg||_{L^1} \leq ||f||_{L^p}||g||_{L^q}.$$

Theorem 2.5. If $0 < \mu(X) < \infty$ and $p_0 \leq p_1$, then $L^{p_1}(X) \subset L^{p_0}(X)$ and

$$\frac{1}{\mu(X)^{1/p_0}}||f||_{L^{p_0}} \leq \frac{1}{\mu(X)^{1/p_1}}||f||_{L^{p_1}}.$$

Proposition 2.6. If $X = \mathbb{Z}$ and μ is the counting measure, then $L^{p_0}(\mathbb{Z}) \subset L^{p_1}(\mathbb{Z})$ if $p_0 \leq p_1$. Moreover,

$$||f||_{L^{p_1}} \leq ||f||_{L^{p_0}}.$$

For $p = \infty$, then $L^\infty(X, \mathcal{M}, \mu)$ consists of all equivalence classes of measurable functions on X, so that there exists $0 < M \in \mathbb{R}$ such that $|f(x)| \leq M$ almost everywhere $[\mu]$ on X. Then, when endowed with the norm

$$||f||_\infty = \inf\{0 < M \in \mathbb{R} : |f(x)| \leq M \text{ for almost every } x \in X\},$$

this collection becomes a Banach space.

Classical results in the theory of L^p spaces are essential in understanding the mechanics of Banach spaces. Results such as the Hahn-Banach theorem, closed graph theorem, and more illustrate the richness of the structure in a Banach space. However, here, we state only one result and illustrate it with an example.

Theorem 2.7. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $p, q \in [1, \infty]$ be Hölder conjugates in the sense that $1/p + 1/q = 1$. Then, for any measurable functions $f \in L^p(\mu)$ and $g \in L^q(\mu)$, we have that

$$||fg||_1 \leq ||f||_p||g||_q.$$

Example 2.3. If we consider the counting measure on $\mathbb{N}$, Hölder's inequality yields that

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/q},$$

for every $x, y \in \{\mathbb{R}^n, \mathbb{C}^n\}$, where p and q are Hölder conjugates.

Chapter 3

Hilbert Spaces

An indespensible tool in the study of PDE theory, Quantum Mechanics, Fourier analysis, and Ergodic theory, Hilbert spaces bear fruit in that they allow for elementary methods of linear algebra and analysis to be generalized from finite-dimensional Euclidean vector spaces to infinite-dimensional linear spaces. Their geometry is rich and it lends for an intuitive understanding of their importance. Here, we will begin by analyzing the geometry of a Hilbert space to then introduce the algebra of operators on Hilbert spaces. Then, we will establish spectral heuristics to introduce spectral measures, integrals, and the celebrated, and extremely important, spectral theorem.

3.1 The Geometry of Hilbert Spaces

Here, we will primarily work with vector spaces over the field $\mathbb{C}$. In case a distinction is made, we will assume to work over $\mathbb{C}$. The simplest of all such vector spaces is $\mathbb{C}$, itself, if we consider it under the ordinary operations of addition and scalar multiplication of complex numbers. In the same vain, as important as $\mathbb{C}$ is among all complex vector spaces, so are the linear transformations whose range space coincides with $\mathbb{C}$ important among all linear transformations. Such transformations are called *linear functionals*, and they will be of importance for us.

Definition 3.1. A *linear functional* ξ on a complex vector space $\mathcal{H}$ is an additive and homogenous linear mapping $\xi : H \to \mathbb{C}$. We call ξ a *conjugate linear functional* if $\xi(\alpha x) = \alpha^* \xi(x)$, for all $\alpha \in \mathbb{C}$ and $x \in H$.

Definition 3.2. A *bilinear functional* on a complex vector space $\mathcal{H}$ is a function $\varphi : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that if

$$\xi_y(x) = \eta_x(y) = \varphi(x, y),$$

then, for every $x, y \in \mathcal{H}$, ξ_y is a linear functional and η_x a conjugate linear functional.

Definition 3.3. The *quadratic form* $\hat{\varphi}$ induced by a bilinear functional φ on a complex vector space $\mathcal{H}$ is a mapping defined by $\hat{\varphi}(x) := \varphi(x, x)$.

Definition 3.4. An *inner product* in a complex vector space $\mathcal{H}$ is a bilinear functional φ such that:

1. For every $x, y \in \mathcal{H}$, then $\varphi(x, y) = \varphi^*(y, x)$.
2. For every $x \in \mathcal{H}$, $\varphi(x, x) > 0$.

We call $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ an *inner product space* and typically denote $\langle\cdot,\cdot\rangle$ as the inner product.

Theorem 3.5. Let $\mathcal{H}$ be an inner product space and define a function $||\cdot|| : H \to \mathbb{C}$ by setting $||x|| := \sqrt{\langle x, x\rangle}$, for all $x \in H$. Then, $||\cdot||$ is a norm on H.

Theorem 3.6. Let $\mathcal{H}$ be an inner product space. If we let $d(x, y) = ||x - y||$, for every $x, y \in H$, then (H, d) is a metric space.

Definition 3.7. A *Hilbert space* is an inner product space which is complete in the metric induced by the inner product.

Given the above definitions, we may now illustrate a few examples of Hilbert spaces.

3.1.1 Examples

Example 3.1. Let $(X, \mathcal{M}, \mu)$ be a measure space and recall L^p space associated to $(X, \mathcal{M}, \mu)$. If $p = 2$, we get a Hilbert space. Indeed, if we let

$$L^2(\mu) := \left\{ f : X \to \mathbb{C} \text{ measurable} \;\middle|\; \int_X |f|^2 \, \mathrm{d}\mu < \infty \right\},$$

and define the hermitian form

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)} \, \mathrm{d}\mu(x),$$

then $(L^2(\mu), \langle\cdot,\cdot\rangle)$ is a Hilbert space. This canonical space will be of use in the later sections.

Example 3.2. Let $k \in \mathbb{Z}^+$ and $\Omega \subset \mathbb{R}^n$. The Sobolev space $H^2(\Omega)$ is the collection of all L^2 functions whose weak derivatives are also L^2. That is,

$$H^2(\Omega) := \left\{ f \in L^2(\Omega) \mid \forall\ |\alpha| \leq k,\ \partial_x^\alpha f \in L^2(\Omega) \right\},$$

for $|\alpha| = \sum_{i=1}^n \alpha_i$, where $\partial_x^\alpha f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f$. Endowed with the norm

$$\langle f, g\rangle_{k,\Omega} = \sum_{|\alpha| \leq k} \langle \partial_x^\alpha f, \partial_x^\alpha g\rangle_{L^2(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega \partial_x^\alpha f \overline{\partial_x^\alpha g} \, \mathrm{d}\mu,$$

$H^2(\Omega)$ becomes a Hilbert space.

Example 3.3. Given two measurable spaces $(X, \mathscr{A})$, $(Y, \mathcal{M})$, a measurable mapping $f : X \to Y$, and a measure $\mu : \mathscr{A} \to [0, \infty]$, the *push-forward* of μ is defined to be the measure $f_*(\mu) : \mathcal{M} \to$

$[0, \infty]$ given by

$$f_*(\mu)(B) = \mu(f^{-1}(B)),$$

for all $B \in \mathcal{M}$. Using this, a natural Lebesgue measure on the unit circle S^1—here thought of as a subset of $\mathbb{C}$—may be defined using a push-forward construction and Lebesgue measure μ on $\mathbb{R}$. Let λ denote the restriction of the Lebesgue measure on $[0, 2\pi)$ and let $f : [0, 2\pi) \to S^1$ be the natural bijection defined by $f(\theta) = e^{i\theta}$. Then, the arising "Lebesgue measure" on S^1 is the push-forward measure $f_*(\lambda)$. Let $f_*(\lambda)$ be denoted by $\mathrm{d}\theta$, as this push-forward measure can be called the "arc length measure" due to the $f_*(\lambda)$-measure of an arc in S^1 yielding precisely its arc length. The definition of the Hardy-Hilbert space utilizes a normalized Lebesgue arc-length measure on the boundary of the open unit disk, as is demonstrated in its definition.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disk. Define the Hardy-Hilbert space, $H^2(\mathbb{D})$, as

$$H^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} \;\middle|\; \sup_{0<r<1} \int |f(re^{i\theta})|^2 \, \frac{\mathrm{d}\theta}{2\pi} < \infty \right\},$$

wherein for $f \in H^2(\mathbb{D})$,

$$||f||^2 = \sup_{0<r<1} \int |f(re^{i\theta})|^2 \, \frac{\mathrm{d}\theta}{2\pi}.$$

Employing the polarization identity yields that $H^2(\mathbb{D})$ is indeed a Hilbert space.

3.2 The Algebra of Operators

Recall that a linear transformation A from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ is *bounded* if there exists $0 < \alpha \in \mathbb{R}$ such that $||Ax|| \leq \alpha ||x||$, for all $x \in \mathcal{H}$. Moreover, the norm of A is

$$||A|| := \inf\{\alpha \in \mathbb{R}^+ : ||Ax|| \leq \alpha ||x||, \text{ for all } x \in \mathcal{H}\}.$$

Theorem 3.8. A linear transformation $A : \mathcal{H} \to \mathcal{K}$ is bounded if, and only if, it is continuous.

In the case when $\mathcal{K} = \mathbb{C}$, there is a powerful result which completely characterizes all bounded linear functionals.

Theorem 3.9. (Riesz representation theorem). A linear functional ξ on $\mathcal{H}$ is bounded if, and only if, there exists a $y \in \mathcal{H}$ such that $\xi(x) = \langle x, y \rangle$ for all $x \in \mathcal{H}$. Such a y, if it exists, is unique.

Before proceeding to the adjoint of an operator, we state useful theorems regarding invertible operators which will be employed later. Recall that an operator A is invertible if there exists an operator B such that $AB = BA = I$. Exploiting the geometry afforded to us in a Hilbert space, it is useful to have geometric conditions for invertibility at our disposal. Note that the range of an operator is always a linear manifold, but it is not necessarily a subspace.

Theorem 3.10. If A is an operator on $\mathcal{H}$ and $\alpha \in \mathbb{R}^+$ such that $||Ax|| \geq \alpha ||x||$ for all $x \in \mathcal{H}$, then the range of A is closed.

Theorem 3.11. An operator A acting on a Hilbert space $\mathcal{H}$ is invertible if, and only if, its range is dense in $\mathcal{H}$ and there exists $\alpha \in \mathbb{R}^+$ such that $||Ax|| \geq \alpha ||x||$, for all $x \in \mathcal{H}$.

3.2.1 The Adjoint

Let us build from the ground up. Take $A \in B(X, Y)$, for X and Y arbitrary Hilbert spaces. Let $y \in Y$ be fixed and define the linear functional $\psi_y : X \to \mathbb{C}$ by $\psi_y(x) = \langle Ax, y \rangle_Y$. Because A is linear, then ψ_y is linear, and hence $\langle \cdot, y \rangle_Y$ is a linear form. Notice that

$$|\psi_y(x)| \leq ||Ax|| \cdot ||y|| \leq ||A|| \cdot ||x||_X \cdot ||y||_Y,$$

since A is bounded. Hence, ψ_y is bounded. Therefore, $\psi_y \in X^*$, the dual of X. By the Riesz representation theorem, for every $y \in Y$, there exists a unique $z_y \in X$ such that $\psi_y(x) = \langle x, z_y \rangle_X$, for every $x \in X$. Now, define the operator $A^* : Y \to X$ by $z_y := A^* y$. Then, the defining property of $A*$ is

$$\langle Ax, y \rangle_Y = \langle x, A^* y \rangle_X,$$

for every $x \in X$, $y \in Y$. We call A^* the *adjoint* of A.

Theorem 3.12. If A is an operator on $\mathcal{H}$, and if $\varphi(x, y) = \langle Ax, y \rangle$ for every $x, y \in \mathcal{H}$, then φ is a bounded bilinear functional and $||\varphi|| = ||A||$. If, conversely, φ is a bounded bilinear functional, then there exists a unique operator A such that $\varphi(x, y) = \langle Ax, y \rangle$, for all $x, y \in \mathcal{H}$.

Theorem 3.13. If A is an operator on $\mathcal{H}$, then there exists a unique operator A^*, called the *adjoint* of A, such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$, for all $x, y \in \mathcal{H}$. Moreover, A^* is such that $||A^*|| = ||A||$.

3.2.2 Hermitian and Normal Operators

Observe that **Theorem 3.12** can be used to show that if A is a Hermitian operator, then

$$||A|| = \sup_{||x||=1} |\langle Ax, x \rangle|.$$

Indeed, recall that $||A|| := \sup_{||x|| \leq 1, x \in \mathscr{H}} ||Ax||$. Then

$$\sup_{||x||=1} |\langle Ax, x \rangle| \leq \sup_{||x||=1} (||Ax|| \cdot ||x||) \leq \sup_{||x||=1} \left(||A|| \cdot ||x||^2 \right) = ||A||.$$

To obtain the opposite inequality, observe that if x is any non-zero vector in $\mathscr{H}$, then $x/||x||$ is a unit vector, so that

$$\left\langle A \frac{x}{||x||}, \frac{x}{||x||} \right\rangle \leq \sup_{||x||=1} |\langle Ax, x \rangle|,$$

and hence for all $x \in \mathscr{H}$

$$\langle Ax, x\rangle \leq \sup_{||x||=1} |\langle Ax, x\rangle| \cdot |||x||^2.$$

Now, select any $x, y \in \mathscr{H}$ such that $||x|| = ||y|| = 1$. Then, by the Hermitian property of A, we have that

$$\begin{aligned}\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle &= 2\langle Ax, y\rangle + 2\langle Ay, x\rangle \\ &= 2\langle Ax, y\rangle + 2\langle y, Ax\rangle \\ &= 4\ \mathrm{Re}\left(\langle Ax, y\rangle\right).\end{aligned}$$

By (1) and by the Parallelogram Law, we have

$$\begin{aligned}4\ \mathrm{Re}\left(\langle Ax, y\rangle\right) &\leq |\langle A(x+y), x+y\rangle| + |\langle A(x-y), x-y\rangle \\ &\leq \sup_{||x||=1} |\langle Ax, x\rangle| \cdot ||x+y||^2 + \sup_{||x||=1} |\langle Ax, x\rangle| \cdot ||x-y||^2 \\ &= 2 \sup_{||x||=1} |\langle Ax, x\rangle| \left(||x||^2 + ||y||^2\right) \\ &= 4 \sup_{||x||=1} |\langle Ax, x\rangle|.\end{aligned}$$

Hence, $\mathrm{Re}\left(\langle Ax, y\rangle\right) \leq \sup_{||x||=1} |\langle Ax, x\rangle|$, for all unit vectors x and y. Let $\langle Ax, y\rangle = |\langle Ax, y\rangle| \exp i\theta$. Then, $ye^{i\theta}$ is simply another unit vector, so that

$$\sup_{||x||=1} |\langle Ax, x\rangle| \geq \mathrm{Re}\left(\langle Ax, e^{-i\theta}y\rangle\right) = \mathrm{Re}\left(e^{i\theta}\langle Ax, y\rangle\right) = |\langle Ax, y\rangle|.$$

Hence,

$$||Ax|| = \sup_{||y||=1} |\langle Ax, y\rangle| \leq \sup_{||x||=1} |\langle Ax, x\rangle|.$$

Because this is true for all unit vectors $x \in \mathscr{H}$, then

$$||A|| \leq \sup_{||x||=1} |\langle Ax, x\rangle|,$$

and hence the result follows.

Definition 3.14. An operator A is *Hermitian* if $A = A^*$.

If A is an operator, then there exist two uniquely determined Hermitian operators A_1 and A_2 such that $A = A_1 + iA_2$. Because the real and imaginary parts of an operator often fail to commute, we observe that operator theory requires a lot more care than does the corresponding theory of complex numbers. We need a definition for which this pathology does not occur. Here it is:

Definition 3.15. An operator A is *normal* if A commutes with A^*.

For us, there is a very special class of normal operators, U, which satisfy the relation $UU^* = U^*U = 1$. These such operators are called *unitary*. Unitary operators are characterized as those invertible operators whose inverse is equal to its adjoint. Moreover, they are of importance because the unitary operators on a Hilbert space $\mathcal{H}$ are exactly the automorphisms of $\mathcal{H}$.

3.3 Spectral Heuristics

Definition 3.16. Let A be an operator on a Hilbert space, $\mathcal{H}$. The *spectrum* of A, denoted $\Lambda(A)$, is defined as

$$\Lambda(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I_{\mathcal{H}} \text{ is not invertible}\}.$$

that in the finite-dimensional case, a necessary and sufficient condition for the invertibility of an operator is the vanishing of its determinant. However, this makes no sense in the not necessarily finite dimensional case. This motivates a definition of the spectrum of an operator on any space. In the finite-dimensional case, because the determinant of $(A - \lambda) \in \mathbb{C}[\lambda]$, a polynomial in the indeterminate λ, whose zeros are exactly the proper values of A, it follows that in this case the spectrum of A is exactly the set of its proper values. By way of illustration, let us consider the following examples.

Example 3.4. Let Ω be a compact Hausdorff space and consider the Banach algebra of functions $C(\Omega)$. Then, for any $\phi \in C(\Omega)$, we have that $\Lambda(\phi) = \phi(\Omega)$. To wit, let $\lambda \in \Lambda(\phi)$, for some $\phi \in C(\Omega)$. Then, $(\lambda - \phi) \notin \text{Inv}(C(\Omega))$, the group of invertible elements of $C(\Omega)$. That is, $1/(\lambda - \phi) \notin C(\Omega)$, so it must be that $(\lambda - \phi)(x) = 0$ for some $x \in \Omega$. Hence, $f(x) = \lambda$, so that $\lambda \in \phi(\Omega)$. Therefore, $\Lambda(\phi) \subseteq \phi(\Omega)$.

Conversely, if $\lambda \in \phi(\Omega)$, then there exists an $x \in \Omega$ such that $\lambda = \phi(x) \in \phi(\Omega)$. Hence, $(\lambda - \phi)(x) = 0$, and so $(\lambda - \phi)(\Omega) \ni 0$ for $x \in \Omega$. Therefore, $1/(\lambda - \phi) \notin C(\Omega)$, so that $(\lambda - \phi) \notin \text{Inv}(C(\Omega))$. Hence, $\lambda \in \Lambda(\phi)$, and consequently $\phi(\Omega) \subseteq \Lambda(\phi)$.

Example 3.5. Let X be a non-empty set and consider $\ell^\infty(X)$. Then, for any $\psi \in \ell^\infty(X)$, $\Lambda(\psi) = \overline{\psi(X)}$, the closure in $\mathbb{C}$. Indeed, select $\psi \in \ell^\infty(X)$ and suppose $\lambda \notin \overline{\psi(X)}$. Then, there exists an $\epsilon > 0$ such that for every $x \in X$, then $|\psi(x) - \lambda| \geq \epsilon$. Define $\xi : X \to \mathbb{C}$ by $x \mapsto 1/(\psi(x) - \lambda)$. Notice that for every $x \in X$

$$|\xi(x)| = \frac{1}{|\psi(x) - \lambda|} < \frac{1}{\epsilon}.$$

Hence, $\xi \in \ell^\infty(X)$. Moreover, a small computation shows that

$$\xi(\psi - \lambda)(x) = (\psi - \lambda)\xi(x) = 1,$$

for all $x \in X$, so that $\xi := \psi^{-1}$. Hence, $(f - \lambda) \in \text{Inv}(\ell^\infty(X))$, and so $\lambda \notin \Lambda(\psi)$. That is, $\Lambda(\psi) \subseteq \overline{\psi(X)}$.

Conversely, suppose $\lambda \in \overline{\psi(X)}$. For every $\epsilon > 0$, there exists $x \in X$ so that $|\psi(x) - \lambda| < \epsilon$. Consequently, ξ is not well-defined or its norm in $\ell^\infty(S)$ is that of

$$||\xi||_\infty = \sup_{x \in X} \frac{1}{|\psi(x) - \lambda|} > \frac{1}{\epsilon}.$$

As $\epsilon > 0$ was arbitrary, the ξ is not bounded, so $\xi \notin \ell^{\infty(X)}$. Hence, $(\psi - \lambda)$ is not invertible in $\ell^\infty(X)$, so $\lambda \in \Lambda(\psi)$. Thus, $\overline{\psi(X)} \subseteq \Lambda(\psi)$.

Before proceeding to spectral analysis, let us digress an consider a more elementary theory as an illustration [1]. Recall that a real-valued, bounded, measurable function f on a finite measure space Ω can be uniformly approximated by simple functions. That is, for every $\epsilon > 0$, there exists a finite, disjoint collection $\{\chi_i\}_i$ of measurable characteristic functions, and a finite collection $\{\lambda_i\}_i \subset \mathbb{R}$ such that for all $\omega \in \Omega$

$$\left| f(\omega) - \sum_{i \in I} \lambda_i \chi_i(\omega) \right| < \epsilon.$$

If, for any Borel set $M \subset \mathbb{R}$, we set $E(M) := \chi_{f^{-1}(M)}$, the above is simply

$$\left| f - \sum_{i \in I} \lambda_i E(M_i) \right| < \epsilon,$$

where $\{M_i\}_i$ is a finite partition of $[\alpha, \beta] \ni f(\omega)$, for all $\omega \in \Omega$, such that for every $i \in I$, we may select $\lambda_i \in M_i$. Notice that the sum and its summands look similar to those occurring in the theory of integration— certainly for Lebesgue. The map E is a sort of measure—which we will define later—associating a characteristic function on Ω with each Borel subset of $\mathbb{R}$. Because for every $i \in I$, we select $\lambda_i \in M_i$ of a partition of $[\alpha, \beta]$, the integral that appears to be in the background is of the form

$$\int \lambda \, \mathrm{d}E(\lambda).$$

We will make sense of this integral in the proceeding sections.

3.3.1 Spectral Measures and Spectral Integrals

Before proceeding, recall that a *projection* on a subspace $\mathcal{N} \subset \mathcal{H}$ is the transformation P defined, for every $z \in \mathcal{H}$ of the form $x + y$, where $x \in \mathcal{N}$ and $y \in \mathcal{N}^\perp$, by $Pz = x$. In this case,

$$\mathcal{N} = P(\mathcal{N}) \oplus \mathrm{Ker}(P).$$

Definition 3.17. The projection P on a subspace $\mathcal{N}$ is an idempotent and Hermitian operator. If $\mathcal{N}$ is the non-trivial subspace, then $||P|| = 1$.

If X is an arbitrary set with a Boolean σ-algebra $\mathcal{S}$, a *spectral measure* in X is a function E mapping $\mathcal{S}$ to idempotent, Hermitian operators defined on $\mathcal{H}$ such that $E(X) = 1$ and where for a disjoint sequence $\{M_n\}_n \subseteq \mathcal{S}$, then

$$E\left(\bigcup_n M_n\right) = \sum_n E(M_n).$$

A great example to keep in mind is found by letting X be a measure space with measure μ, considering $L^2(\mu)$, and setting

$$E(M)\phi = \chi_M \phi,$$

for every measurable $M \in \mathcal{S}$ and all $\phi \in L^2(\mu)$. Elementary techniques of measure theory show us that E is, indeed, a spectral measure. Let us establish conditions necessary for projection-valued functions to be spectral measures.

Theorem 3.18. A projection-valued function E on the σ-algebra $\mathcal{S}$ of measurable subsets of a measurable space X is a spectral measure if, and only if, $E(X) = 1$ and for every $x, y \in \mathcal{H}$, the complex-valued set function μ defined for every $M \in \mathcal{S}$ by $\mu(M) = \langle E(M)x, y\rangle$ is countably additive.

For $(X, \mathcal{S})$ measurable space, let $\mathcal{B}(X)$ denote the set of all bounded, complex-valued, measurable functions on X.

Theorem 3.19. Let $(X, \mathcal{S})$ be a measurable space. If E is a spectral measure in X and if $f \in \mathcal{B}(X)$, then there exists a unique operator A such that

$$\langle Ax, y\rangle = \int f(\lambda)\ \mathrm{d}\langle E(\lambda)x, y\rangle$$

for all $x, y \in \mathcal{H}$.

3.3.2 The Spectral Theorem

The analogs of bounded, real-valued, measurable functions in the theory of Hilbert spaces are Hermitian operators. Because a function is the characteristic function of a set if, and only if, it is idempotent, then—algebraically— the analogs of characteristic functions are projections. The approximability of arbitrary functions by simple ones corresponds to the approximability of Hermitian operators by finite, real, linear combinations of idempotent, Hermitian operators. However, why is this useful for us? The usefulness of this development lies in its purpose: providing a tool for understanding complicated objects in terms of simpler ones. We will be able to recapture a Hermitian operator by constructing a spectral measure which, in turn, will reflect properties of the given operator. We will encounter a beautiful example in Chapter 5 when considering a peculiar shift operator on $\ell^2(\mathbb{Z})$. For now, let us state the grand Spectral theorem and various versions found in [2] which will be utilized in this manuscript.

Theorem 3.20 (Hermitian Operator). If A is a Hermitian operator, then there exists a unique, real, compact complex spectral measure E, such that

$$A = \int \lambda\ \mathrm{d}E(\lambda).$$

Theorem 3.21 (**Normal Operator**). If A is a Normal operator, then there exists a unique, compact complex spectral measure E, such that

$$A = \int \lambda \, \mathrm{d}E(\lambda).$$

Before stating the other versions of the Spectral theorem, recall that for a fixed Hermitian operator A, and $\omega \in \mathcal{H}$, then $\xi : C(\Lambda(A)) \to \mathbb{C}$ defined by

$$\xi(f) = \langle f(A)\omega, \omega \rangle$$

is a positive linear functional.

Theorem 3.22 (**Riesz Markov**). Let Ω be a locally compact Hausdorff space and ϕ a positive linear functional on $C_c(\Omega)$, the space of compactly supported, complex-valued, continuous functions on Ω. Then, there exists a unique, positive Borel measure μ on Ω such that

$$\phi(f) = \int_\Omega f(\Omega) \, \mathrm{d}\mu(\omega),$$

for every $\omega \in \Omega$.

Applying the above theorem to ξ, there exists a unique measure μ_ω on the compact set $\Lambda(A)$ with the property that

$$\langle f(A)\omega, \omega \rangle = \int_{\Lambda(A)} f(\lambda) \, \mathrm{d}\mu_\omega.$$

The measure μ_ω is the spectral measure associated with $\omega \in \mathcal{H}$.

Definition 3.23. A vector $\xi \in \mathcal{H}$ is called *cyclic* for A if finite linear combinations of elements $\{(A^n\xi)_n\}_{n=0}^{\infty}$ are dense in $\mathcal{H}$.

Lemma 3.24. Let A be a bounded, Hermitian operator with cyclic vector ξ. Then, there is a unitary operator $U : \mathcal{H} \to L^2(\Lambda(A), \mathrm{d}\mu_\xi)$ with

$$(UAU^{-1}f)(\lambda) = \lambda f(\lambda),$$

with equality of elements in the L^2-sense.

Lemma 3.25. Let A be a Hermitian operator on a separable Hilbert space, H. Then, there is a direct sum decomposition

$$H = \bigoplus_{n=1}^{N} H_n$$

with $N = 1, 2, \ldots,$ or ∞ so that:

1. $\psi \in H_n$ implies $A\psi \in H_n$ (H_n is invariant under A).

2. For every n, there is a $\phi_n \in H_n$ which is cyclic for $A|H_n$. That is,

$$H_n = \overline{\{f(A)\phi_n \mid f \in C(\Lambda(A))\}}.$$

Theorem 3.26. Let A be a bounded, Hermitian operator on a separable Hilbert space, H. Then, there exists measures $\{\mu_n\}_{n=1}^{\infty}$ on $\Lambda(A)$ and a unitary operator

$$U : H \to \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}, \mathrm{d}\mu_n)$$

so that

$$(UAU^{-1}\psi)_n(\lambda) = \lambda\psi_n(\lambda),$$

where $\psi = \langle \psi_1(\lambda), \ldots \rangle \in \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}, \mathrm{d}\mu_n)$.

Corollary 3.27. Let A be a Hermitian operator on a separable Hilbert space, $\mathcal{H}$. Then, there exists a finite measure space (X, μ), an essentially bounded function F on X, a unitary map $U : \mathcal{H} \to L^2(X, \mathrm{d}\mu)$ so that

$$(UAU^{-1}f)(x) = F(x)f(x) = (M_F f)(x).$$

Chapter 4

Algebras

Given a field, an algebra defined on this field is simply a vector space together with a bilinear product. This algebraic structure provides a lot of robust, and rich results, which illustrate how well-behaved these spaces are. In this chapter, we will illustrate operator algebras, Banach algebras, and C^*-algebras to establish a result that is the culmination of the background material required in developing our analysis in chapter 5.

4.1 Banach Algebras

Definition 4.1. An *algebra* A is a vector space over a field F, together with a bilinear map $* : A \times A \to A$ defined by $(a, b) \mapsto ab$ such that

$$a * (bc) = (ab) * c,$$

for every $a, b, c \in A$.

Definition 4.2. Let A be an algebra. If we endow A with a norm that is sub-multiplicative in the sense that for every $a, b \in A$, then $||ab|| \leq ||a||||b||$, then we say $(A, || \cdot ||)$ is a *normed algebra*.

Definition 4.3. A complete normed algebra is called a *Banach algebra*.

In the case that A admits a unit element of norm one, then we will call A a unital normed algebra. To illustrate, consider the following example.

Example 4.1. Let $\ell^1(\mathbb{Z})$ be the linear space of all complex-valued functions defined on $\mathbb{Z}$ that are absolutely summable. For $f, g \in \ell^1(\mathbb{Z})$, define the *convolution* map $* : \mathbb{Z} \to \mathbb{C}$ by

$$(f * g)(k) = \sum_{n \in \mathbb{Z}} f(k - n)g(n).$$

Considering $\ell^1(\mathbb{Z})$ with ring multiplication given by convolution, then $\ell^1(\mathbb{Z})$ is an Algebra over $\mathbb{C}$. Indeed, for $\lambda, \gamma \in \mathbb{C}$ and $f, g, h \in \ell^1(\mathbb{Z})$, then $(\lambda f + \gamma g) \in \ell^1(\mathbb{Z})$, so

$$\begin{aligned}
[(\lambda f + \gamma g) * h](k) &= \sum_{n\in\mathbb{Z}} (\lambda f + \gamma g)(k-n)h(n) \\
&= \sum_{n\in\mathbb{Z}} (\lambda f(k-n) + \gamma g(k-n))\, h(n) \\
&= \lambda \sum_{n\in\mathbb{Z}} f(k-n)h(n) + \gamma \sum_{n\in\mathbb{Z}} g(k-n)h(n) \\
&= \lambda(f*h)(k) + \gamma(g*h)(k),
\end{aligned}$$

and

$$\begin{aligned}
[f * (\lambda g + \gamma h)(k)] &= \sum_{n\in\mathbb{Z}} f(k-n)(\lambda g + \gamma h)(n) \\
&= \sum_{n\in\mathbb{Z}} f(k-n)\,(\lambda g(n) + \gamma h(n)) \\
&= \lambda \sum_{n\in\mathbb{Z}} f(k-n)g(n) + \gamma \sum_{n\in\mathbb{Z}} f(k-n)h(n) \\
&= \lambda(f*g)(k) + \gamma(f*h)(k),
\end{aligned}$$

holds for every $k \in \mathbb{Z}$. Moreover, endowed with the usual pointwise operations and norm

$$||f|| = \sum_{n\in\mathbb{Z}} |f(n)|, \quad \text{for all } f \in \ell^1(\mathbb{Z}),$$

then this is a Banach space. Because $f \in \ell^1(\mathbb{Z})$, the above sum exists. Notice that

$$\begin{aligned}
||f*g|| &= \sum_{n\in\mathbb{Z}} |(f*g)(n)| \\
&= \sum_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} f(n-k)g(k) \right| \\
&\leq \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |f(n-k)| \cdot |g(k) \\
&= \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |f(n-k)| \cdot |g(k)| \\
&= \sum_{k\in\mathbb{Z}} \left(|g(k)| \sum_{n\in\mathbb{Z}} |f(n-k)| \right) \\
&= \sum_{k\in\mathbb{Z}} |g(k)| ||f|| \\
&= ||f|| \cdot ||g||.
\end{aligned}$$

Hence, $f*g \in \ell^1(\mathbb{Z})$. Moreover, because $\mathbb{Z}$ is abelian under addition, then with unit $\chi_{\{0\}}$, it follows that $\ell^1(\mathbb{Z})$ is an abelian, unital Banach algebra with ring multiplication given by convolution.

Example 4.2. The algebra $M_n(\mathbb{C})$ of square matrices with entries in $\mathbb{C}$ is identified with the collection of all bounded linear operators on $\mathbb{C}^n$, $B(\mathbb{C}^n)$. Hence, it is a unital Banach algebra.

If we consider all upper triangular matrices of the form

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \dots & \lambda_{1n} \\ 0 & \lambda_{22} & \dots & \dots & \lambda_{2n} \\ 0 & 0 & \lambda_{33} & \dots & \lambda_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_{nn} \end{bmatrix}$$

then these matrices form a subalgebra of $M_n(\mathbb{C})$. That is, this set of matrices, $U_n(\mathbb{C})$, is a subspace of $M_n(\mathbb{C})$ and for any $U_1, U_2 \in U_n(\mathbb{C})$, then $U_1U_2 \in U_n(\mathbb{C})$.

4.2 C^*-Algebras

Definition 4.4. Let A be an algebra. An *involution* is a conjugate-linear map $a \mapsto a^*$ on A such that $a^{**} = a$ and $(ab)^* = b^*a^*$, for every $a, b \in A$. The tuple $(A, *)$ is called a $*$-algebra.

Definition 4.5. A *Banach $*$-algebra* is a $*$-algebra A endowed with a complete, submultiplicative norm such that for any $a \in A$, then $||a^*|| = ||a||$.

Definition 4.6. A *C^*-algebra* is a Banach $*$-algebra such that for any $a \in A$, then

$$||a^*a|| = ||a||^2.$$

As pointed out in [3], the C^* identity in the above definition is very strong. In fact, much more is known about the struture of algebras that satisfy that identity that perhaps any other non-trivial algebras. Because of this involution, the theory of C^*-algebras is often motivated as the study of real analysis in infinite dimensions.

Example 4.3. Here is an example we are all familiar with. Take $\mathbb{C}$ with involution given by complex conjugation. This scalar field is a prime example of a C^*-algebra.

Example 4.4. If Ω is a locally compact Hausdorff space, then the set of all complex-valued functions on Ω that vanish at infinity, $C_0(\Omega)$, is a C^*-algebra with involution given by complex conjugation.

Example 4.5. If $\mathcal{H}$ is a Hilbert space, then the set of all bounded, linear operators on $\mathcal{H}$, $B(\mathcal{H})$ is a C^*-algebra with involution given by the adjoint operation. In fact, every C^*-algebra can be realized as a C^*-subalgebra of some $B(\mathcal{H})$. This is the context of the Gelfand-Neimark theorem!

Before concluding, let us provide one more example illustrating the beauty in this structure. Let Ω be a compact Hausdorff space and let μ be a positive, regular Borel measure on Ω. That is, a positive measure for which every μ-measurable set can be approximated by open, μ-measurable sets from above and my compact μ-measurable sets from below. For $\varphi \in L^\infty(\Omega, \mu)$, define the multiplication operator $M_\varphi : L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ by $f \mapsto \varphi f$, for every $f \in L^2(\Omega, \mu)$. Notice that M_φ is bounded. Indeed,

$$||M_\varphi f||_2^2 = \int_\Omega |\varphi f|^2 \, \mathrm{d}\mu \le ||\varphi||_\infty^2 \int_\Omega |f|^2 \, \mathrm{d}\mu,$$

which implies that $||M_\varphi|| \le ||\varphi||_\infty$. The map

$$\Psi : L^\infty(\Omega, \mu) \to B(L^2(\Omega, \mu))$$

given by $\Psi\varphi = M_\varphi$ is a $*$-homomorphism of C^*-algebras. That is, Ψ is a homomorphism of $*$-algebras that preserves adjoints. In particular, M_φ is normal, whereby $M_\varphi^* = M_{\overline{\varphi}}$. As we will show in the spectral analysis in chapter 5, these operators are typical of all normal operators.

Is $\mathscr{B}(\Omega)$ is the σ-algebra of all Borel subsets of Ω, and if $S \in \mathscr{B}(\Omega)$, then χ_S is a projection in $L^\infty(\Omega, \mu)$. Hence, $E(S) = M_{\chi_S}$ is a projection operator in $B(L^2(\Omega, \mu))$. In fact, the map $E : \mathscr{B}(\Omega) \to B(L^2(\Omega, \mu))$ is a spectral measure relative to the tuple $(\Omega, L^2(\Omega, \mu))$. More can be said, though. It can be shown that $||M_\varphi|| = ||\varphi||_\infty$, and therefore the map Ψ is an isometric $*$-isomorphism of $L^\infty(\Omega, \mu)$ onto a C^*-subalgebra of $B(L^2(\Omega, \mu))$. Thus, as we will exemplify in Example 5.1, $\Lambda(M_\varphi) = \Lambda_{L^\infty}(\varphi)$.

Chapter 5

Composition Operators

We have finally arrived at our investigation. Motivated by the investigation of the topological space of composition operators on the Banach algebra of bounded functions defined on an unbounded, locally finite metric space in the operator norm topology and essential norm topology [4], we further hone in on the properties of a constructed subspace of the considered space. Specifically, we begin by studying the weighted Banach space L^∞_μ via composition operators induced by self-maps of the underlying metric space, which then allows for us to construct a Hilbert subspace of the weighted functional Banach space. In doing so, we obtain operator-theoretic results concerning composition operators, extend the underlying metric space to avail ourselves of the structure, and consider the spectral structure of composition operators.

5.1 Preliminaries

Let X be a Banach algebra of functions on a domain Ω, and $S(\Omega)$ the set of self-maps of Ω. For $\varphi \in S(\Omega)$, the induced linear operator $C_\varphi : X \to X$, defined by

$$C_\varphi f = f \circ \varphi,$$

for all $f \in X$, is called the *composition operator with symbol* φ. This operator was first studied in [5] on the Hardy-Hilbert space $H^2(\mathbb{D})$, wherein $X = H^2(\mathbb{D})$, and $\Omega = \mathbb{D} \subseteq \mathbb{C}$. These are spaces which

$$H^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} \;\middle|\; \sup_{0<r<1} \int |f(re^{i\theta})|^2 \, \frac{\mathrm{d}\theta}{2\pi} < \infty \right\},$$

where $\mathrm{d}\theta$ is the Lebesgue arc-length measure on $\partial\mathbb{D}$, and wherein the norm for $f \in H^2(\mathbb{D})$ is

$$||f||^2 = \sup_{0<r<1} \int |f(re^{i\theta})|^2 \, \frac{\mathrm{d}\theta}{2\pi}.$$

Typically, regarding these operators, $H^2(\mathbb{D})$ and $H^p(\mathbb{D})$ are the first spaces of inquiries. However, here, we begin our investigation on a functional Banach space of complex-valued functions.

Let (T, d) be an unbounded, locally finite metric space with distinguished element o. That is, for any pair $M, N > 0$, there exists $v \in T$ with $|v| := d(o, v) \geq M$ and where the set $\{v \in T \mid |v| < N\} \subseteq T$ is finite. Notice that because T is unbounded, then it admits sequences whose distance from o is strictly increasing in $\mathbb{Z}_{\geq 0}$. Moreover, if f is a self-map of T with finite range, then there exists a positive constant M for which $|f(v)| \leq M$ for every $v \in T$; we denote this by set by $S_F(T)$. Otherwise, $f \in S_I(T)$, if f has infinite range.

Let $\mu : T \to \mathbb{R}^+$ be a positive function on T, which we will call a *weight* on T. Define the linear space of bounded functions on T, $L^\infty_\mu(T)$, by

$$L^\infty_\mu(T) = \left\{ f : T \to \mathbb{C} \;\Big|\; \sup_{v \in T} \mu(v)|f(v)| < \infty \right\}.$$

Theorem 5.1. The linear space $L^\infty_\mu(T)$ is a complex Banach space under

$$||f||_\mu = \sup_{v \in T} \mu(v)|f(v)|.$$

Proof. Let $f, g \in L^\infty_\mu$. It is clear that $||f||_\mu \geq 0$ and $||f||_\mu = 0$ if and only if $f \equiv 0$ on T. Moreover, for any $\alpha \in \mathbb{C}$, then

$$||\alpha f||_\mu = \sup_{v \in T} \mu(v)|(\alpha f)(v)| = \sup_{v \in T} \mu(v)|\alpha f(v)| = \sup_{v \in T} \mu(v)|\alpha| \cdot |f(v)| = |\alpha| \cdot ||f||_\mu,$$

and

$$\begin{aligned} ||f + g||_\mu &= \sup_{v \in T} \mu(v)|(f + g)(v)| \\ &= \sup_{v \in T} \mu(v)|f(v) + g(v)| \\ &\leq \sup_{v \in T} \mu(v) \left(|f(v)| + |g(v)|\right) \\ &= \sup_{v \in T} \mu(v)|f(v)| + \sup_{v \in T} \mu(v)|g(v)| \\ &= ||f||_\mu + ||g||_\mu. \end{aligned}$$

Hence, $||\cdot||_\mu$ is indeed a norm. Let $\epsilon > 0$ and fix $w \in T$. Suppose $\{f_n\}$ is a Cauchy sequence in L^∞_μ. By definition, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have that $||f_n - f_m||_\mu < \epsilon\mu(w)$. Because μ is positive, then

$$|f_n(w) - f_m(w)| = \left| \frac{\mu(w)(f_n(w) - f_m(w))}{\mu(w)} \right| \leq \frac{1}{\mu(w)} \left(\sup_{w \in T} \mu(w)|f_n(w) - f_m(w)| \right) < \epsilon.$$

That is, $\{f_n(w)\}$ is Cauchy in $\mathbb{C}$ and converges to $f(w) \in \mathbb{C}$. We can select $w \in T$ arbitrarily so that $f_n \to f$ pointwise on T. Therefore, for a fixed $w \in T$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, then $|f(w) - f_n(w)| < 1/\mu(w)$. Applying the triangle inequality yields that

$$\mu(w)|f(w) - f_n(w)| < 1 \implies \mu(w)|f(w)| < 1 + \mu(w)|f_n(w)|.$$

Hence,

$$\sup_{w\in T}\mu(w)|f(w)| < \sup_{w\in T}(1+\mu(w)|f_n(w)|) \implies ||f||_\mu < 1+||f_n||_\mu.$$

Because $\{f_n\}$ is Cauchy, it is bounded, so that $||f||_\mu$ is finite and hence $f \in L^\infty_\mu$. To complete the proof, we must show that the Cauchy sequence $\{f_n\}$ converges to f in norm on L^∞_μ. Aiming for a contradiction, suppose that there exists $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $||f_{n_k}-f||_\mu \geq \epsilon$, for all $k \in \mathbb{N}$. For all such k, select a point $v_{n_k} \in T$ such that $|f_{n_k}(v_{n_k})-f(v_{n_k})| \geq \epsilon/\mu(v_{n_k})$. Then, as $\{f_{n_k}\}$ is a subsequence of a Cauchy sequence, it is Cauchy in L^∞_μ, so that there exists $M \in \mathbb{N}$ such that for all $j, l \geq M$, then $||f_{n_j} - f_{n_l}||_\mu < \epsilon/2$. Moreover, as

$$\mu(v)|f_{n_j}(v) - f_{n_l}(v)| \leq \sup_{v\in T}\mu(v)|f_{n_j}(v) - f_{n_l}(v)| = ||f_{n_j} - f_{n_l}||_\mu,$$

then, in particular for some $q \geq M$, it follows that

$$\mu(v_{n_M})|f_{n_M}(v_{n_M}) - f_{n_q}(v_{n_M})| < \epsilon/2.$$

Because $f_{n_q} \to f$ pointwise, for sufficiently large $q \in \mathbb{N}$, we notice that

$$\mu(v_{n_M})|f_{n_q}(v_{n_M}) - f(v_{n_M})| < \epsilon/2.$$

Therefore,

$$\mu(v_{n_M})|f_{n_M}(v_{n_M}) - f(v_{n_M})| \leq \mu(v_{n_M})|f_{n_M}(v_{n_M}) - f_{n_q}(v_{n_M})| + \mu(v_{n_M})|f_{n_q}(v_{n_M}) - f(v_{n_M})| < \epsilon,$$

which is clearly a contradiction on our choice $v_{n_M} \in T$. Thus, it must be that $||f_n - f||_\mu \to 0$ as $n \to \infty$. Therefore, L^∞_μ is complete under $||\cdot||_\mu$. □

It is clear that $L^\infty_\mu(T)$ under pointwise operations is an algebra over $\mathbb{C}$. In fact, it is a Banach algebra. Indeed, notice that for $f, g \in L^\infty_\mu(T)$

$$||fg||_\mu = \sup_{v\in T}\mu(v)|f(v)g(v)| \leq \sup_{v\in T}\mu(v)|f(v)| \cdot \sup_{v\in T}\mu(v)|g(v)| = ||f||_\mu||g||_\mu.$$

Definition 5.2. A Banach space X of complex-valued functions defined on a set Ω is a *functional* Banach space if for every $x \in \Omega$, the point evaluation functional $K_x f = f(x)$ is bounded.

Proposition 5.3. The Banach algebra $(L^\infty_\mu, ||\cdot||_\mu)$ is a Functional Banach space [6].

Proof. Fix $v \in T$. Select $f \in L^\infty_\mu$ such that $||f||_\mu = 1$. Then,

$$|f(v)| = \frac{\mu(v)}{\mu(v)} \cdot |f(v)| \leq \frac{1}{\mu(v)}\sup_{v\in T}\mu(v)|f(v)| = \frac{||f||_\mu}{\mu(v)} < \infty,$$

holds for all $v \in T$, as μ is positive. By definition, the result follows. □

5.2 Construction of a Hilbert Subspace

The investigation of composition operators on $L^\infty(T)$ in [7] is very interesting, and the results are plentiful. However, availing ourselves of the theory in chapter 4, we would like to consider composition operators on spaces which have a richer geometry, and which we can exploit to develop more results of the class of composition operators we are considering here. Firstly, let us establish our first result.

Theorem 5.4. Let $\ell^2_\mu(T)$ be defined by

$$\ell^2_\mu(T) = \left\{ \phi : T \to \mathbb{C} \;\middle|\; \sum_{v \in T} \mu(v)|\phi(v)|^2 < \infty \right\}.$$

Endowing this space with a Hermitian form given by

$$\langle \phi, \psi \rangle_\mu = \sum_{v \in T} \mu(v)\phi(v)\overline{\psi(v)}, \quad \text{for} \quad \phi, \psi \in \ell^2_\mu(T),$$

then $\ell^2_\mu(T)$ is a Hilbert subspace of the Banach algebra $L^\infty_\mu(T)$.

Proof. Firstly, we will show that $\ell^2_\mu(T)$ is a subspace of $L^\infty_\mu(T)$. Let $\psi : T \to \mathbb{C}$ be defined by $v \mapsto 0 + 0i$. Clearly, $\psi \in \ell^2_\mu(T)$. Moreover, let $\varphi, \phi, \xi \in \ell^2_\mu(T)$ and $\alpha \in \mathbb{C}$. Then,

$$\sum_{v \in T} \mu(v)|\alpha\phi(v)|^2 = |\alpha|^2 \sum_{v \in T} \mu(v)|\phi(v)|^2 < \infty.$$

Additionally,

$$\begin{aligned}
\sum_{v \in T} \mu(v)|(\xi + \varphi)(v)|^2 &= \sum_{v \in T} \mu(v)|\xi(v) + \varphi(v)|^2 \\
&\leq \sum_{v \in T} \mu(v)\left(|\xi(v)| + |\varphi(v)|\right)^2 \\
&= \sum_{v \in T} \mu(v)\left(|\xi(v)|^2 + 2|\xi(v)||\varphi(v)| + |\varphi(v)|^2\right) \\
&= \sum_{v \in T} \mu(v)|\xi(v)|^2 + 2\sum_{v \in T} \mu(v)|\xi(v)\varphi(v)| + \sum_{v \in T} \mu(v)|\varphi(v)|^2.
\end{aligned}$$

Applying Hölder's inequality (**Theorem 2.7**) yields that

$$\sum_{v \in T} \mu(v)|\xi(v)\varphi(v)| \leq \left(\sum_{v \in T} \mu(v)|\xi(v)|^2 \right)^{1/2} \left(\sum_{v \in T} \mu(v)|\varphi(v)|^2 \right)^{1/2} < \infty,$$

and thus $\ell^2_\mu(T)$ is a subspace.

Next, we show that the form $\langle \cdot, \cdot \rangle_\mu : \ell^2_\mu(T) \times \ell^2_\mu(T) \to \mathbb{R}^+$ defined as above is indeed an inner product. Notice

$$\langle \phi, \phi \rangle_\mu = \sum_{v \in T} \mu(v)\phi(v)\overline{\phi(v)} = \sum_{v \in T} \mu(v)|\phi(v)|^2 \geq 0,$$

as μ is positive on T. It is clear that $\langle \phi, \phi \rangle_\mu = 0$ if, and only if, $\phi \equiv 0$ on T. Furthermore,

$$\langle \psi, \xi \rangle_\mu = \sum_{v \in T} \mu(v)\psi(v)\overline{\xi(v)} = \sum_{v \in T} \mu(v)\overline{\overline{\psi(v)}} \cdot \overline{\xi(v)} = \sum_{v \in T} \mu(v)\overline{\xi(v)\overline{\psi(v)}} = \overline{\langle \xi, \psi \rangle}_\mu.$$

Letting $\alpha, \beta \in \mathbb{C}$, and $\xi, \varphi \in \ell^2_\mu(T)$, then $(\alpha\psi + \beta\xi) \in \ell^2_\mu(T)$, so that

$$\begin{aligned}
\langle \alpha\psi + \beta\xi, \varphi \rangle_\mu &= \sum_{v \in T} \mu(v)(\alpha\psi + \beta\xi)(v)\overline{\varphi(v)} \\
&= \sum_{v \in T} \mu(v) \left(\alpha\psi(v) + \beta\xi(v)\right) \overline{\varphi(v)} \\
&= \sum_{v \in T} \left(\alpha\mu(v)\psi(v) + \beta\mu(v)\xi(v)\right) \overline{\varphi(v)} \\
&= \alpha \sum_{v \in T} \mu(v)\psi(v)\overline{\varphi(v)} + \beta \sum_{v \in T} \mu(v)\xi(v)\overline{\varphi(v)} \\
&= \alpha\langle \psi, \varphi \rangle_\mu + \beta\langle \xi, \varphi \rangle_\mu.
\end{aligned}$$

Finally, we show that the inner product space $\ell^2_\mu(T)$ is a Hilbert space. Let $\{\phi_n\}$ be a Cauchy sequence in $\ell^2_\mu(T)$ so that as $m, n \to \infty$, then

$$\sum_{v \in T} \mu(v)|\phi_m(v) - \phi_n(v)|^2 \to 0.$$

For all $v \in T$, the sequence $\{\phi_n(v)\}$ is Cauchy in $\mathbb{C}$, and hence ϕ_n converges to some ϕ. We will show that $\phi_n \to \phi$ in the $\ell^2_\mu(T)$ metric induced by the inner product and that $\phi \in \ell^2_\mu(T)$. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $||\phi_m(v) - \phi_n(v)||^2_\mu \leq \epsilon$. Letting $m \to \infty$ yields that $||\phi - \phi_n||_\mu \leq \epsilon$, and hence $\phi_n \to \phi$ on T in the induced metric. Because $\phi_n \in \ell^2_\mu(T)$ for all $n \in \mathbb{N}$, then there exists $K_n \in \mathbb{R}^+$ such that

$$||\phi_n||_\mu = \left(\sum_{v \in T} \mu(v)|\phi_n(v)|^2 \right)^{1/2} < K_n^{1/2}.$$

Letting $\epsilon > 0$, it follows that

$$\begin{aligned}
||\phi||_\mu &= \sqrt{\sum_{v \in T} \mu(v)|\phi(v)|^2} \\
&= \sqrt{\sum_{v \in T} \mu(v)|\phi(v) - \phi_n(v) + \phi_n(v)|^2} \\
&\leq \sqrt{\sum_{v \in T} \mu(v)\big(|\phi(v) - \phi_n(v)| + |\phi_n(v)|\big)^2} \\
&= \sqrt{\sum_{v \in T} \mu(v)|\phi(v) - \phi_n(v)|^2 + 2\sum_{v \in T} \mu(v)|\phi_n(v)[\phi(v) - \phi_n(v)]| + \sum_{v \in T} \mu(v)|\phi_n(v)|^2} \\
&\leq \sqrt{\sum_{v \in T} \mu(v)|\phi(v) - \phi_n(v)|^2 + 2\left(\sum_{v \in T} \mu(v)|\phi_n(v)|^2\right)^{1/2} \left(\sum_{v \in T} \mu(v)|\phi(v) - \phi_n(v)|^2\right)^{1/2} + \sum_{v \in T} \mu(v)|\phi_n(v)|^2} \\
&< \sqrt{\epsilon(\epsilon + 1) + K_n^{1/2}(2 + K_n^{1/2})} < \infty.
\end{aligned}$$

That is, $||\phi||_\mu < \infty$, and thus $\phi \in \ell^2_\mu(T)$. Hence, $(\ell^2_\mu(T), \langle\cdot,\cdot\rangle_\mu)$ is a Hilbert subspace of $L^\infty_\mu(T)$. □

For $\varphi \in S(T)$, we define C_φ on $\ell^2_\mu(T)$ just as we did for the Banach algebra $L^\infty_\mu(T)$. Linearity of C_φ is clear, so we begin our investigation of the boundedness of this operator.

5.3 Boundedness of C_φ

Here, we investigate when C_φ is an element of the C^*-algebra $B(\ell^2_\mu(T))$. Set

$$\sigma_\varphi = \sum_{v\in T} \frac{\mu(v)}{\mu(\varphi(v))}.$$

Theorem 5.5. Let $\varphi \in S(T)$. If, $\sigma_\varphi < \infty$ then $C_\varphi \in B(\ell^2_\mu(T))$.

Proof. Let $\varphi \in S_I(T)$ and let $R(\varphi)$ denote its range. Moreover, suppose μ is a finite measure on T. If $\sigma_\varphi < \infty$, then for any $f \in \ell^2_\mu(T)$, it follows that

$$\begin{aligned}
||C_\varphi f||^2_\mu &= \sum_{v\in T} \mu(v)|f(\varphi(v))|^2 \\
&= \sum_{v_i\in R(\varphi)} \sum_{v\in\varphi^{-1}(\{v_i\})} \mu(v)|f(\varphi(v))|^2 \\
&= \sum_{v_i\in R(\varphi)} \sum_{v\in\varphi^{-1}(\{v_i\})} \mu(v)|f(v_i)|^2 \\
&= \sum_{v_i\in R(\varphi)} \sum_{v\in\varphi^{-1}(\{v_i\})} \frac{\mu(v)}{\mu(\varphi(v))}\mu(v_i)|f(v_i)|^2 \\
&= \sum_{v_i\in R(\varphi)} \left[\mu(v_i)|f(v_i)|^2 \sum_{v\in\varphi^{-1}(\{v_i\})} \frac{\mu(v)}{\mu(\varphi(v))}\right] \\
&\leq \sum_{v_i\in R(\varphi)} \mu(v_i)|f(v_i)|^2\sigma_\varphi \\
&= \sigma_\varphi \sum_{v_i\in R(\varphi)} \mu(v_i)|f(v_i)|^2 \\
&\leq \sigma_\varphi||f||^2_\mu.
\end{aligned}$$

Hence, $C_\varphi \in B(\ell^2_\mu(T))$. □

Notice that the converse of the above theorem need not be true. Consider the following example.

Example 5.1. Let $\varphi \equiv id_T$. For any $f \in \ell^2_\mu(T)$, we have

$$||C_\varphi f||_\mu = \left(\sum_{v\in T} \mu(v)|f(\varphi(v))|^2\right)^{1/2} = \left(\sum_{v\in T} \mu(v)|f(v)|^2\right)^{1/2} = ||f||_\mu < \infty,$$

so that $C_\varphi \in B(\ell^2_\mu(T))$. However, $\sigma_\varphi = \infty$.

Theorem 5.6. Let $\varphi \in S_F(T)$. If $C_\varphi \in B(\ell^2_\mu(T))$, then $\sum_{v\in T}\mu(v) < \infty$.

Proof. Suppose $C_\varphi \in B(\ell^2_\mu(T))$. Set $f = \chi_{R(\varphi)}$, for $R(\varphi)$ the range of φ. Then,

$$||f||^2_\mu = \sum_{v\in T}\mu(v)|f(v)|^2 = \sum_{v\in T}\mu(v)|\chi_{R(\varphi)}(v)|^2 = \sum_{w\in R(\varphi)}\mu(w) < \infty,$$

so that $||C_\varphi f||^2_\mu < \infty$. That is,

$$||C_\varphi f||^2_\mu = \sum_{v\in T}\mu(v)|f(\varphi(v))|^2 = \sum_{v\in T}\mu(v)|\chi_{R(\varphi)}(\varphi(v))|^2 = \sum_{v\in T}\mu(v) < \infty.$$

□

It is worthy to mention that the converse of the above theorem may fail. Indeed, if we take $\varphi : T \to T$ with finite range and suppose $\sum_{v\in T}\mu(v) < \infty$, then C_φ being an element of the operator algebra $B(\ell^2_\mu(T))$ relies on $\varphi(v) \leq v$ for all $v \in T$. This is all too restrictive. The following theorem remedies this dilemma by providing a necessary condition on σ_φ for φ to induce a bounded composition operator on $\ell^2_\mu(T)$.

Theorem 5.7. Let $\varphi \in S(T)$ be one-to-one. If $\sup_{v\in T}\frac{\mu(v)}{\mu(\varphi(v))} < \infty$, then $C_\varphi \in B(\ell^2_\mu(T))$

Proof. Suppose $\sup_{v\in T}\frac{\mu(v)}{\mu(\varphi(v))} < \infty$. Then, for all $v \in T$, it holds that

$$\mu(v) \leq \left(\sup_{v\in T}\frac{\mu(v)}{\mu(\varphi(v))}\right)\mu(\varphi(v)).$$

Let $f \in \ell^2_\mu(T)$. Then,

$$\begin{aligned}
||C_\varphi f|| &= \left(\sum_{v\in T}\mu(v)|f(\varphi(v))|^2\right)^{1/2} \\
&\leq \left(\sum_{v\in T}\left(\sup_{v\in T}\frac{\mu(v)}{\mu(\varphi(v))}\right)\mu(\varphi(v))|f(\varphi(v))|^2\right)^{1/2} \\
&\leq \left(\sup_{v\in T}\frac{\mu(v)}{\mu(\varphi(v))}\right)^{1/2}\left(\sum_{v\in T}\mu(\varphi(v))|f(\varphi(v))|^2\right)^{1/2} \\
&\leq \sup_{v\in T}\frac{\mu(v)}{\mu(\varphi(v))}\left(\sum_{v\in T}\mu(v)|f(v)|^2\right)^{1/2} \\
&= \sup_{v\in T}\frac{\mu(v)}{\mu(\varphi(v))}||f||_\mu < \infty.
\end{aligned}$$

□

Recall that given any set X, then (X, Σ) is a measurable space if we take the power set of X as the σ-algebra, Σ. Then, the counting measure μ on (X, Σ) is the positive measure defined by

$$\mu(E) = \begin{cases} |E|, & \text{if } E \text{ is finite} \\ +\infty, & \text{if } E \text{ is infinite} \end{cases}$$

for every $E \in \Sigma$. Using this, the following result was inspired by the analysis conducted in [8].

Theorem 5.8. Let $\varphi \in S(T)$ and take μ to be the counting measure. Then, $C_\varphi \in B(\ell^2(T))$ if, and only if, there exists an $M > 0$ such that for all $v \in T$, then $\mu(\varphi^{-1}(\{v\})) < M$.

Proof. Suppose $C_\varphi \in B(\ell^2(T))$ with $||C_\varphi|| = M$. Let $v_0 \in T$ be arbitrary and consider $f = \chi_{\{v_0\}}$. Then, $f \in \ell^2(T)$ with $||f|| = 1$. Because C_φ is bounded as an operator on $\ell^2(T)$, then $C_\varphi f \in \ell^2(T)$. Furthermore, $||C_\varphi f|| \leq ||C_\varphi|| \cdot ||f|| = M$. That is,

$$\sum_{v \in T} |f(\varphi(v))|^2 = \sum_{v \in T} |\chi_{\{v_0\}}(\varphi(v))|^2 = \sum_{v \in \varphi^{-1}(v_0)} |\chi_{\{v_0\}}(\varphi(v))|^2 = |\varphi^{-1}(\{v_0\})| = \mu(\varphi^{-1}(\{v_0\})) \leq M.$$

Conversely, suppose there exists an $M > 0$ such that $\mu(\varphi^{-1}(\{v\})) < M$ holds for all $v \in T$. Select $f \in \ell^2(T)$. Then,

$$||C_\varphi f||^2 = \sum_{v \in T} |f(\varphi(v))|^2 = \sum_{w \in \varphi^{-1}(\{v\})} |f(w)|^2 \mu(\varphi^{-1}(\{v\})) \leq M||f||^2.$$

Hence, $C_\varphi \in B(\ell^2(T))$. □

Theorem 5.9. Let μ be the counting measure on T. If $C_\varphi \in B(\ell^2(T))$, then

$$||C_\varphi||^2 = \inf\{M > 0 \mid \mu(\varphi^{-1}(\{v\})) \leq M, \text{ for all } v \in T\}.$$

Proof. Suppose $C_\varphi \in B(\ell^2(T))$ and let $v \in T$. Let $\{\lambda^v\}_v$ be the sequence defined by $\lambda^v(w) = \delta_{wv}$, the Kronecker delta. Then,

$$\mu(\varphi^{-1}(\{v\})) = \sum_{v \in T} |\delta_{\varphi(v)v}(\varphi(v))|^2 = ||C_\varphi \lambda^v||^2 \leq ||C_\varphi||^2 \cdot ||\lambda^v||^2 = ||C_\varphi||^2.$$

Since this holds for all $v \in T$, then

$$\inf\{M > 0 \mid \mu(\varphi^{-1}(\{v\})) \leq M, \text{ for all } v \in T\} \leq ||C_\varphi||^2.$$

Now, suppose $\mu(\varphi^{-1}(\{v\})) \leq M$ for all $v \in T$. By the previous theorem, $C_\varphi \in B(\ell^2(T))$. Hence

$$||C_\varphi||^2 \leq \inf\{M > 0 \mid \mu(\varphi^{-1}(\{v\})) \leq M, \text{ for all } v \in T\}.$$

Thus, the result follows. □

More can be said. Indeed, we can obtain injectivity of the self map φ under certain conditions on its induced composition operator. Consider the following theorem.

Theorem 5.10. Let $\varphi \in S(T)$. Then, φ is one-to-one if, and only if, $C_\varphi \in B(\ell^2(T))$ and $||C_\varphi|| = 1$.

Proof. Suppose φ is one-to-one as a self-map of T. Take $f \in \ell^2(T)$. Then,

$$||C_\varphi f||^2 = \sum_{v \in T} |f(\varphi(v))|^2 \mu(\varphi^{-1}(\{v\})) = ||f||^2,$$

as φ is one-to-one. Hence, $C_\varphi \in B(\ell^2(T))$. Moreover, because $\mu(\varphi^{-1}(\{v\})) = 1$, then

$$||C_\varphi||^2 = \inf\{M > 0 \mid \mu(\varphi^{-1}(\{v\})) \leq M, \text{ for all } v \in T\} = 1,$$

and hence $||C_\varphi|| = 1$. Conversely, suppose C_φ a bounded operator of norm one on $\ell^2(T)$. That is, $||C_\varphi|| = 1$, and hence $\mu(\varphi^{-1}(\{v\})) = 1$, so that φ is one-to-one. □

5.4 Invertibility and the Adjoint of C_φ

As stated and proved in [9], on $L^p(\lambda)$, necessary and sufficient conditions that C_φ be invertible are the invertibility of φ and the inducibility of the composition operator C_φ^{-1} by φ^{-1}, where λ is a σ-finite measure on a standard Borel space. In our case, as will be shown, the invertibility of φ alone is necessary and sufficient for the invertibility of the composition operator it induces [8]. To that effect, identify T with $\mathbb{N}$ and consider the measure space $(\mathbb{N}, \Sigma, \mu)$, where μ is the counting measure. Recall that $B(X)$ is the space of all linear, bounded operators defined on X. Henceforth, the omission of μ in $\ell^2_\mu(T)$ indicates that we are considering the space with the counting measure.

Lemma 5.11. Let $C_\varphi \in B(\ell^2(T))$. Then, C_φ is invertible if, and only if, φ is invertible.

Proof. Let $C_\varphi \in B(\ell^2(T))$ and suppose φ is invertible. Then, there exists a self-map ψ of T such that $\varphi \circ \psi = \psi \circ \varphi = id$. Because ψ is one-to-one, $C_\psi \in B(\ell^2(T))$. Notice

$$C_\psi(C_\varphi(f)) = C_\psi(f(\varphi(v))) = f(\varphi(\psi(v))) = f(v),$$

for all $f \in \ell^2(T)$. Therefore, C_φ is invertible in $B(\ell^2(T))$, with inverse $(C_\varphi)^{-1} = C_{\varphi^{-1}}$.

Conversely, suppose C_φ is invertible. If φ is not one-to-one, then $\varphi(v) = \varphi(w)$, for some $v \neq w$, and—moreover— $f(v) = f(w)$ for all $f \in C_\varphi(\ell^2(T))$. Hence, C_φ is not onto, a contradiction. If φ is not onto, then there exists $k \in T$ such that $k \notin \varphi(T)$. Then,

$$C_\varphi \lambda^k(v) = C_\varphi \delta_{vk} = \delta_{\varphi(v)k} = 0.$$

Thus, C_φ is not one-to-one, a contradiction. □

Theorem 5.12. Let $C_\varphi \in B(\ell^2(T))$. Then, C_φ is invertible if, and only if, C_φ is unitary.

Proof. Suppose C_φ is invertible. For $f \in \ell^2(T)$, let $R(f)$ and $R(C_\varphi f)$ be the ranges of f and $C_\varphi f$, respectively. By the invertibility of $C_\varphi f$, then φ is invertible, and hence $R(f) = R(C_\varphi f)$. Therefore,

$$||C_\varphi f|| = \left(\sum_{v\in T} |f(\varphi(v))|^2\right)^{1/2} = \left(\sum_{v\in T} |f(v)|^2\right)^{1/2} = ||f||,$$

so that C_φ is unitary. Conversely, if C_φ is unitary, then $C_\varphi^* C_\varphi = C_\varphi C_\varphi^* = I$. Hence, C_φ is invertible. □

For the sake of the results to come, let us digress a bit. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces over a field $F \in \{\mathbb{R}, \mathbb{C}\}$. Recall that every linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ defines a Hermitian adjoint $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ with the defining property that for every $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$,

$$\langle Ax, y\rangle_{\mathcal{H}_2} = \langle x, A^* y\rangle_{\mathcal{H}_1}.$$

Now, Let $\varphi \in S(T)$ such that $C_\varphi \in B(\ell^2(T))$. Consider the transformation $\Psi : \ell^2(T) \to \ell^2(T)$ given by

$$\Psi f(v) = \begin{cases} \langle f, C_\varphi \lambda^v\rangle = \sum\limits_{w\in\varphi^{-1}(\{v\})} f(w), & \varphi^{-1}(\{v\}) \neq \emptyset \\ 0, & \varphi^{-1}(\{v\}) = \emptyset \end{cases}.$$

Notice that the invertibility of φ implies that $\Psi f(v) = f(\varphi^{-1}(\{v\})$, which is well-defined. Moreover, it is clear that

$$\Psi f(v) = \langle f, \chi_{\varphi^{-1}(\{v\})}\rangle.$$

We claim that Ψ is the adjoint of C_φ. Indeed, notice that for any $f, g \in \ell^2(T)$, if $\varphi^{-1}(\{v\})$ is non-empty, then

$$\langle C_\varphi f, g\rangle = \sum_{v\in T} f(\varphi(v))\overline{g(v)} = \sum_{v\in R(\varphi)} \left[f(v) \sum_{w\in\varphi^{-1}(\{v\})} \overline{g(w)} \right] = \langle f, \Psi g\rangle.$$

If $\varphi^{-1}(\{v\})$ is empty,

$$\langle C_\varphi f, g\rangle = \sum_{v\in T} f(\varphi(v))\overline{g(v)} = 0 = \langle f, \Psi g\rangle.$$

Corollary 5.13. Let $C_\varphi \in B(\ell^2(T))$. Then, C_φ^* is a composition operator if, and only if, C_φ is invertible.

Proof. Suppose C_φ is invertible. Then, C_φ is unitary, so that $C_\varphi^* C_\varphi = C_\varphi C_\varphi^* = I$, and therefore $C_\varphi^* = (C_\varphi)^{-1} = C_{\varphi^{-1}}$. That is, C_φ^* is the composition operator induced by φ^{-1}. Conversely, suppose C_φ^* is a composition operator. By definition, $C_\varphi^* = C_\psi$, for some $\psi \in S(T)$. Then, for all $v \in T$

$$\chi_{\{\varphi(v)\}} = C_\varphi^* \chi_{\{v\}} = C_\psi \chi_{\{v\}} = \chi_{\psi^{-1}(\{v\})}.$$

Therefore, $\{\varphi(v)\} = \psi^{-1}(\{v\})$, so that ψ is invertible. Hence, C_ψ is invertible. Thus, C_φ is invertible. □

We end this section with a en equivalence of statements that culminates the developments of this section.

Theorem 5.14. Suppose $C_\varphi \in \ell^2_\mu(T)$. Then, the following are equivalent:

i. φ is invertible.

ii. C_φ is invertible.

iii. C_φ is unitary.

Proof. The equivalence of i. and ii. is the context of the Lemma 5.11. If φ is invertible, then the above corollary implies that $C_\varphi^* = C_{\varphi^{-1}}$, and hence C_φ is unitary. Now, if C_φ is unitary, then $C_\varphi^* = C_\varphi^{-1}$, the composition operator induced by φ^{-1}. By the Lemma in section 3, it follows that φ is invertible. □

5.5 Spectral Results

In this section, we utilize the results from chapter 3 to provide two avenues of investigation that will allow us to characterize the spectrum of a bounded composition operator on $\ell^2(T)$ with μ the counting measure. Both avenues rely on obtaining a unitarily equivalent operator on some L^2 space, and availing ourselves of the Hilbert space structure it affords to obtain a characterization of the spectrum of C_φ via the unitarily equivalent operator. To illustrate the technique, let us consider an example.

Example 5.2. Consider $\ell^2(\mathbb{Z})$ equipped with the counting measure. Let $A : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be defined by $A(x_n)_n = (x_{n-1} + x_{n+1})_n$. It can be shown that

$$\Lambda(A) = [-2, 2].$$

However, this heavily relies on the nice structure of $\ell^2(\mathbb{Z})$ and the fact that A is simply the sum of a left and right shift operator. Let us apply the above corollary to obtain a unitarily equivalent operator and find the spectral structure of A via a multiplication operator. Let $(y_n)_n \in \ell^2(\mathbb{Z})$.

Then,

$$\begin{aligned}||A(y_n)_n||^2 &= \sum_{n\in\mathbb{Z}} |y_{n-1}+y_{n+1}|^2 \\ &\le \sum_{n\in\mathbb{Z}} (|y_{n-1}|+|y_{n+1}|)^2 \\ &\le \sum_{n\in\mathbb{Z}} |y_{n-1}|^2 + 2\left(\sum_{n\in\mathbb{Z}} |y_{n-1}|^2\right)^{1/2}\left(\sum_{n\in\mathbb{Z}} |y_{n+1}|^2\right)^{1/2} + \sum_{n\in\mathbb{Z}} |y_{n+1}|^2 \\ &< \infty,\end{aligned}$$

so that $A \in B(\ell^2(\mathbb{Z}))$. Moreover, A is Hermitian. Indeed,

$$\begin{aligned}\langle A(x_n)_n, (y_n)_n\rangle &= \langle (x_{n-1}+x_{n+1})_n, (y_n)_n)\rangle \\ &= \sum_{n\in\mathbb{Z}} x_n \overline{y_{n-1}+y_{n+1}} \\ &= \langle (x_n)_n, (y_{n-1}+y_{n+1})_n\rangle \\ &= \langle (x_n)_n, A(y_n)_n\rangle.\end{aligned}$$

Let S^1 be the 1-sphere and define μ to be the normalized Lebesgue measure on S^1, with σ-algebra $\mathscr{B}(S^1)$ as the collection of all borel sets on S^1. Notice that $(S^1, \mathscr{B}(S^1), \mu)$ is finite, and—in fact—$\mu(S^1) = 1$. Define $\psi_n : S^1 \to S^1$ by $\psi_n(e^{i\theta}) = e^{ni\theta}$. Then, $\{\psi_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(S^1, \mu)$. Let $U : L^2(S^1, \mu) \to \ell^2(\mathbb{Z})$ be defined by

$$Uf = \left(\int_{S^1} f\overline{\psi_n}\, \mathrm{d}\mu\right)_n.$$

Notice that for any $f \in L^2(S^1, \mu)$,

$$\langle Uf, (x_n)_n\rangle_{\ell^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \left(\int_{S^1} f\overline{\psi_n}\, \mathrm{d}\mu\right)_n \overline{x_n} = \int_{S^1} f\left(\overline{\sum_{n\in\mathbb{Z}} \psi_n x_n}\right) \mathrm{d}\mu = \langle f, \sum_{n\in\mathbb{Z}} \psi_n x_n\rangle_{L^2(S^1)}.$$

That is, $U^*(x_n)_n = \sum_{n\in\mathbb{Z}} \psi_n x_n$. Because $\{\psi_n\}_{n\in\mathbb{Z}}$ is an orthonormal collection, and $\mu(S^1) = 1$, then

$$UU^*(x_n)_n = U\sum_{n\in\mathbb{Z}} \psi_n x_n = \left(\int_{S^1}\left(\sum_{n\in\mathbb{Z}} \psi_n x_n\right)\overline{\psi_m}\, \mathrm{d}\mu\right)_m = (x_n)_n.$$

Moreover, because $f \in L^2(S^1, \mu)$, then

$$U^*Uf = U^*\left(\int_{S^1} f\overline{\psi_n}\, \mathrm{d}\mu\right)_n = \sum_{n\in\mathbb{Z}} \psi_n \left(\int_{S^1} f\overline{\psi_n}\, \mathrm{d}\mu\right)_n = \sum_{n\in\mathbb{Z}} \psi_n \langle f, \psi_n\rangle_{L^2(S^1)} = f.$$

Therefore, U is a unitary operator. Let $\phi : S^1 \to \mathbb{R}$ be such that $\phi(e^{i\theta}) = 2\cos(\theta)$. Because $|\phi| \le 2$ on S^1, then $\phi \in L^\infty(S^1, \mu)$. Moreover, since $2\cos(\theta) = e^{-i\theta} + e^{i\theta}$, then $\phi = \psi_{-1} + \psi_1$.

Finally,

$$\begin{aligned}
UM_\phi\xi &= U(\psi_{-1}\xi + \psi_1\xi)\\
&= \left(\int_{S^1} (\psi_{-1}\xi + \psi_1\xi)\overline{\psi_n}\ \mathrm{d}\mu\right)_n\\
&= \left(\int_{S^1} \xi(\psi_{-1}\overline{\psi_n} + \psi_1\overline{\psi_n})\ \mathrm{d}\mu\right)_n\\
&= \left(\int_{S^1} \xi(\overline{\psi}_{n-1} + \overline{\psi}_{n+1}\ \mathrm{d}\mu\right)_n\\
&= A\left(\int_{S^1} \xi\overline{\psi_n}\ \mathrm{d}\mu\right)_n\\
&= AU\xi.
\end{aligned}$$

That is, $UM_\phi = AU$, so that $A = UM_\phi U^{-1} = UM_\phi U^*$. As a multiplication operator on L^2 induced by an essentially bounded function ϕ, the spectrum of M_ϕ is simply the essential range of ϕ. Because $S^1 \subseteq \mathbb{R}^n$, and μ is the normalized Lebesgue measure, then the essential range of ϕ is the closure of the image of S^1 under ϕ. Hence,

$$\Lambda(M_\phi) = \overline{\phi(S^1)} = [-2, 2].$$

Unitarily equivalent operators have the same spectral structure, so we obtain a description of $\Lambda(A)$ via the above representation.

We can go even further than that. Availing ourselves of the Spectral theorem for Hermitian operators, we can represent A as an integral of the coordinate function against a spectral measure over the spectrum $\Lambda(A)$ of A. To that effect, let $f \in C(\Lambda(A))$. In $L^2(S^1,\mu)$, $f(A)$ is simply multiplication by $f(\phi(e^{i\theta}))$, where $\phi(e^{i\theta}) = 2\cos(\theta)$. Given $\varphi \in L^2(S^1,\mu)$ defined by

$$\varphi = \sum_{n\in\mathbb{Z}} \psi_n x_n,$$

we can define a corresponding sequence $(\zeta_n)_n \in \ell^2(\mathbb{Z})$ by

$$(\zeta_n)_n = \left(\int_{S^1} \varphi\overline{\psi_n}\ \mathrm{d}\mu\right)_n.$$

Then,

$$\begin{aligned}
\int_{\Lambda(A)} f(A)\ \mathrm{d}\mu_{\zeta_n} &= \langle f(A)(\zeta_n)_n, (\zeta_n)_n\rangle_{\ell^2(\mathbb{Z})}\\
&= \langle f(A)UU^{-1}(\zeta_n)_n, UU^{-1}(\zeta_n)_n\rangle_{\ell^2(\mathbb{Z})}\\
&= \langle \left(U^{-1}f(A)U\right) U^{-1}(\zeta_n)_n, U^{-1}(\zeta_n)_n\rangle_{L^2(S^1)}\\
&= \langle M_{f(\phi)}\varphi, \varphi\rangle_{L^2(S^1)}\\
&= \langle f(\phi)\varphi, \varphi\rangle_{L^2(S^1)}\\
&= \int_{S^1} f(2\cos(\theta))|\varphi(\theta)|^2 \frac{\mathrm{d}\theta}{2\pi}.
\end{aligned}$$

In splitting the region of integration symmetrically, and utilizing the change of variables $\lambda = 2\cos(\theta)$, it follows that

$$\int_{\Lambda(A)} f(A)\, \mathrm{d}\mu_{\zeta_n} = \int_{-2}^{2} f(\lambda) \left[\left| \varphi\left(\cos^{-1}\left(\frac{\lambda}{2}\right)\right)\right|^2 + \left| \varphi\left(-\cos^{-1}\left(2\pi + \frac{\lambda}{2}\right)\right)\right|^2 \right] \frac{\mathrm{d}\lambda}{\pi\sqrt{4-\lambda^2}}.$$

That is,

$$\mathrm{d}\mu_{\zeta_n} = \frac{\mathrm{d}\lambda}{\pi\sqrt{4-\lambda^2}},$$

the spectral measure of A.

The ingredients necessary for our method illustrated in the example above inform us of the usefulness of this avenue of investigation. We now introduce the first avenue as a segway into the second.

5.5.1 Multiplication Operator

Recall the multiplication operator described in chapter 4. We established operator theoretic results concerning the operator norm and its spectrum. However, we did not prove the latter. Let us do that here, and in doing so, we will illustrate why this may be fruitful in describing $\Lambda_{\ell^2(T)}(C_\varphi)$.

Example 5.3. Let $(X, \mathcal{M}, \mu)$ be a measure space and fix $\varphi : X \to Y$ an essentially bounded, measurable function. Define the multiplication operator $A_\varphi : L^2(X, \mathcal{M}, \mu) \to L^2(X, \mathcal{M}, \mu)$ by

$$A_\varphi f(x) = \varphi(x) f(x),$$

for $f \in L^\infty(X, \mathcal{M}, \mu)$. Define the spectrum of A_φ, denoted $\Lambda(A_\varphi)$, by

$$\Lambda(A_\varphi) = \{\lambda \in \mathbb{C} \mid (A_\varphi - \lambda) \text{ is not invertible}\}.$$

If we identify $(Y, \mathcal{T})$ with $\mathbb{C}$ by equipping it with its usual topology, the essential range of φ, denoted $\mathcal{R}[\varphi]$, is given by

$$\mathcal{R}[\varphi] = \{\lambda \in \mathbb{C} \mid \forall \epsilon > 0,\ \mu\left(\{x \in X \ :\ |\varphi(x) - \lambda| < \epsilon\}\right) > 0\}$$

We claim that $\Lambda(A_\varphi) = \mathcal{R}[\varphi]$. Indeed, let $\lambda \in \mathbb{C}$ and suppose $\lambda \notin \mathcal{R}[\varphi]$. Then,

$$\mu\left(\{x \in X \ :\ |\varphi(x) - \lambda| < \epsilon\}\right) = 0,$$

for some $\epsilon > 0$. Therefore, $\varphi(x) - \lambda \neq 0$ a.e. on X. Hence, $A_{1/(\varphi-\lambda)}$ is bounded. Because

$$A_{1/(\varphi-\lambda)}\left(\left(A_\varphi - \lambda\right) f\right)(x) = A_{1/(\varphi-\lambda)}\left(\varphi(x) f(x) - \lambda f(x)\right) = \frac{\varphi(x) f(x) - \lambda f(x)}{\varphi(x) - \lambda} = f(x),$$

and similarly,

$$(A_\varphi - \lambda)\left(A_{1/(\varphi-\lambda)}f\right)(x) = (A_\varphi - \lambda)\left(\frac{f(x)}{\varphi(x)-\lambda}\right) = \frac{\varphi(x)f(x) - \lambda f(x)}{\varphi(x)-\lambda} = f(x),$$

then $(A_\varphi - \lambda)$ is invertible, so that $\lambda \notin \Lambda(A_\varphi)$. Thus, $\Lambda(A_\varphi) \subseteq \mathcal{R}[\varphi]$. Conversely, let $\lambda \in \mathcal{R}[\varphi]$. For any $n \in \mathbb{N}$, define

$$\Gamma_n := \{x \in X \ : \ |\varphi(x) - \lambda| < 2^{-n}\}.$$

Select a sequence $\{\Omega_n\}$ such that for all n, $\Omega_n \subseteq \Gamma_n$, where $0 < \mu(\Omega_n) < \infty$. Let $\Psi_n = \chi_{\Omega_n}$, the characteristic function on Ω_n. Then,

$$||(A_\varphi - \lambda)\Psi_n||^2 = \int_{\Omega_n} |\varphi(x) - \lambda|^2 |\Psi_n|^2 \, \mathrm{d}\mu(x) \leq \left(2^{-n}\right)^2 ||\Psi_n||^2 = \frac{||\Psi_n||^2}{2^{2n}},$$

so that the linear transformation $(A_\varphi - \lambda)^{-1}$ is unbounded. Hence, $(A_\varphi - \lambda)$ is not invertible. Therefore, $\lambda \in \Lambda(A_\varphi)$, so that $\mathcal{R}[\varphi] \subseteq \Lambda(A_\varphi)$. Thus, $\Lambda(A_\varphi) = \mathcal{R}[\varphi]$.

We want to describe the structure of $\Lambda_{\ell^2(T)}(C_\varphi)$. However, describing the spectrum of an operator might be difficult. Recall Theorem 3.26 and Corollary 3.27. If we obtain the right space, the right essentially bounded function, and a unitary operator, we can represent a composition operator on $\ell^2_\mu(T)$ as a multiplication operator on an L^2 space. The difficulty lies in finding all the pieces necessary for our construction of a unitary equivalence. As further work, we hope to continue this line of investigation and believe it will be fruitful. The interested reader is referred to [6], where more on this line of inquiry can be found.

5.5.2 Weighted Composition Operator

The above results allow us to establish C_φ acting on $\ell^2_\mu(T)$ as a unitarily equivalent representation of a multiplication operator on some L^2 space induced by an L^∞ function. However, the difficulty comes in finding all the pieces necessary for this representation, which might not have closed forms. In our case, there is no clear way of doing this procedure yet, but further investigation may lead us down the path of obtaining it. Therefore, we turn to another method to to investigate the spectral structure of composition operators. Firstly, let us consider $T = \mathbb{N} \subseteq \mathbb{Z}$ and $\varphi : \mathbb{N} \to \mathbb{Z}$. If we define a weighted composition operator on $\ell^2_\mu(\mathbb{N})$ by

$$W_{\mu,\varphi}f(v) = \begin{cases} \frac{\mu(v)}{\mu(\varphi(v))} f(\varphi(v)), & v \in \mathbb{N} \\ 0, & v \in \mathbb{Z} \setminus \mathbb{N} \end{cases}$$

where $\mu : \mathbb{N} \to \mathbb{R}^+$, then it will be unitarily equivalent to a composition operator C_φ on $L^2(\Omega, \nu)$, where $\Omega = \mathbb{N} \cup R(\varphi)$ and ν is some measure on the σ-algebra of the power set of Ω. Let us consider the follwing definitions found in [10].

Definition 5.15. For $n \in \Omega$, define the *orbit of φ containing n* by

$$O[\varphi; n] = \{k \in \mathbb{Z} \mid \exists i, j \geq 0 \text{ such that } \varphi^i(k) = \varphi^j(n)\}.$$

Definition 5.16. Let $\varphi : \mathbb{N} \to \mathbb{Z}$. If $k \in \mathbb{N}$ and $\varphi^j(k) = k$ for some $j \geq 1$, then the *cycle of* φ *containing* k is

$$C(k) = \{n \in \mathbb{Z} \mid \varphi^t(k) = n, \text{ for some } t \geq 0\}.$$

The map φ is said to have no cycles if $C(k) = \emptyset$, for all $k \in \mathbb{N}$.

Theorem 5.17. Let $\varphi : \mathbb{N} \to \mathbb{Z}$ have one orbit and no cycle, μ a weight on $\mathbb{N}$, and $\Omega = \mathbb{N} \cup R(\varphi)$. Then, the weighted composition operator $W_{\mu,\varphi}$ on $\ell^2_\mu(\mathbb{N})$ is unitarily equivalent to the composition operator C_φ on $L^2(\Omega, \nu)$, for ν a measure on the power set of Ω.

Proof. Define the measure ν by $\nu(0) = 1$ and, for any $w \in \mathbb{N}$, by

$$\nu(w) = \left[\prod_{i=0}^{k-1} \mu(\varphi^i(0)) \left(\prod_{i=0}^{n-1} \mu(\varphi^i(w))\right)^{-1}\right]^{-2} = \left[\mu(0) \dots \mu(\varphi^{k-1}(0)) \left(\mu(w) \dots \mu(\varphi^{n-1}(w))\right)^{-1}\right]^{-2}$$

with the convention that $\varphi^0 = id_{\mathbb{N}}$, and such that $\varphi^n(w) = \varphi^k(0)$, for some $k, n \geq 0$. Define $U : \ell^2_\mu(\mathbb{N}) \to L^2(\Omega, \nu)$ by $f \mapsto f/\sqrt{\nu}$, for all $f \in \ell^2_\mu(\mathbb{N})$. For $f \in \ell^2_\mu(\mathbb{N})$, notice

$$||Uf||^2_\nu = \int_\Omega \left|\frac{f}{\sqrt{\nu}}\right|^2 \,\mathrm{d}\nu = \int_\Omega \nu^{-1}|f|^2 \,\mathrm{d}\nu = \int_\Omega |f|^2 \,\mathrm{d}\mu = ||f||^2_\mu.$$

Now, let us select $h \in L^2(\Omega, \nu)$. Then,

$$\langle Uf, h\rangle_\nu = \langle \nu^{-1/2} f, h\rangle_\nu = \int_\Omega \nu^{-1/2} f\overline{h} \,\mathrm{d}\nu = \int_\Omega f\nu^{1/2}\overline{h} \,\mathrm{d}\mu = \langle f, \nu^{1/2}h\rangle = \langle f, U^{-1}h\rangle,$$

so that $||U|| = 1$ and $U^* = U^{-1}$, implying that U is unitary. If $v \in \mathbb{N}$, then

$$\begin{aligned}
U^* C_\varphi U f(v) &= U^* C_\varphi((\nu^{-1/2} f)(v)) \\
&= U^*((\nu \circ \varphi)^{-1/2}(f \circ \varphi)(v))) \\
&= \nu^{1/2}(\nu \circ \varphi)^{-1/2}(v) f(\varphi(v)) \\
&= \left(\frac{\prod_{i=0}^{k-1} \mu(\varphi^i(0)) \left[\mu(\varphi(v)) \dots \mu(\varphi^{n-2}(\varphi(v)))\right]^{-1}}{\prod_{i=0}^{k-1} \mu(\varphi^i(0)) \left[\mu(\varphi(v)) \dots \mu(\varphi^{n-1}(\varphi(v)))\right]^{-1}}\right) f(\varphi(v)) \\
&= \frac{\mu(v)}{\mu(\varphi(v))} f(\varphi(v)) \\
&= W_{\mu,\varphi} f(v).
\end{aligned}$$

Otherwise, $v \notin \mathbb{N}$, so that $W_{\mu,\varphi} f(v) = 0$ for all $f \in \ell^2_\mu(T)$, and also

$$U^* C_\varphi U f(v) = U^* C_\varphi((\nu^{-1/2} f)(v)) = 0.$$

Thus, C_φ on $L^2(\Omega, \nu)$ is unitarily equivalent to $W_{\mu,\varphi}$ on $\ell^2_\mu(T)$. □

Notice that this result establishes something even more precise than the multiplication operator result. This result allows us to translate the spectral structure of a class of weighted composition operators to the spectral information of composition operators afforded on an L^2 space. This exemplifies another avenue of investigation that will be fruitful in further work to come.

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