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## GEOMETRY OF JUMP SYSTEMS

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**ABSTRACT.** A jump system is a set of lattice points satisfying a certain “two-step” axiom. We present a variety of results concerning the geometry of these objects, including a characterization of two-dimensional jump systems, necessary (though not sufficient) properties of higher-dimensional jump systems, and a characterization of constant-sum jump systems.

**1. Introduction.** A jump system is a set of lattice points that satisfy a simple “two-step” axiom. They were introduced by Bouchet and Cunningham [1] in order to simultaneously generalize delta-matroids (hence matroids) and degree sequences of subgraphs.

Fix a finite set  $S$ . We consider elements of  $\mathbf{Z}^S$  together with the 1-norm  $|x| = \sum_{i \in S} |x_i|$  and the corresponding distance  $d(x, y) = |x - y|$ .

For elements  $x, y \in \mathbf{Z}^S$ , we say  $z \in \mathbf{Z}^S$  is a *step from  $x$  toward (in the direction of)  $y$*  if  $|z - x| = 1$  and  $|z - y| < |x - y|$ . Note that if  $z$  is a step from  $x$  toward  $y$ , then  $z = x \pm e_i$  for some standard unit vector  $e_i$ . For notational convenience, we will use  $x \xrightarrow{y} z$  to denote a step from  $x$  to  $z$  in the direction of  $y$ .

Given a collection of points  $J \subseteq \mathbf{Z}^S$ , we say that  $J$  is a *jump system* if it satisfies Axiom 1.1.

**Axiom 1.1** (2-step axiom). *If  $x, y \in J$  and  $x \xrightarrow{y} z$  with  $z \notin J$ , then there exists  $z' \in J$  with  $z \xrightarrow{y} z'$ .*

The following well-known operations all preserve Axiom 1.1, see [1, 3, 4, 5]. They allow us to simplify many of the later proofs concerning various properties of jump systems.

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Let  $J$  be a jump system. Let  $a \in \mathbf{Z}^S$ . We call  $\{x + a : x \in J\}$  the *translation* of  $J$  by  $a$ . Let  $N \subseteq S$ . We call  $\{x' : x \in J, x'_j = x_j \text{ for } j \notin N, x'_j = -x_j \text{ for } j \in N\}$  the *reflection* of  $J$  in  $N$ . We call  $\{x' \in \mathbf{Z}^{(S \setminus N) \cup \{0\}} : x \in J, x'_j = x_j \text{ for } j \notin N, x'_0 = \sum_{j \in N} x_j\}$  the *reduction* of  $J$  by  $N$ . Let  $J_1$  and  $J_2$  be jump systems on  $\mathbf{Z}^S$ . We call  $\{x + y : x \in J_1, y \in J_2\}$  the *sum* of  $J_1$  and  $J_2$ .

**Example 1.1.** Let  $J_1 = \{(2, 2), (2, 3)\}$ ,  $J_2 = \{(0, 0), (1, 0), (3, 0)\}$ . The translation of  $J_1$  by  $(2, 4)$  is  $\{(4, 6), (4, 7)\}$ . The reflection of  $J_1$  in  $\{2\}$  is  $\{(2, -2), (2, -3)\}$ . The reduction of  $J_2$  by  $\{1, 2\}$  is  $\{(0), (1), (3)\}$ . The sum of  $J_1$  and  $J_2$  is  $\{(2, 2), (2, 3), (3, 2), (3, 3), (5, 2), (5, 3)\}$ .

Let  $-\infty \leq a_i \leq b_i \leq \infty$  for all  $i \in S$ . The set of points  $\{x \in \mathbf{Z}^S : x_i \in [a_i, b_i]\}$  is called a *box*. The following useful theorem characterizes jump systems.

**Theorem 1.1** (Lovász). *Let  $J$  be a jump system, and let  $B^1 \subseteq B^2 \subseteq \dots \subseteq B^r$  be boxes in  $\mathbf{Z}^S$ . Then there is some point  $x \in J$  such that  $x$  is simultaneously of minimal distance to  $B^1, B^2, \dots, B^r$ .*

For  $v$  an element of  $\mathbf{R}^S$ , define  $\bar{v} \in \{-1, 0, 1\}^S$  by

$$\bar{v}_i = \begin{cases} 1 & v_i > 0 \\ 0 & v_i = 0 \\ -1 & v_i < 0 \end{cases}.$$

We will first consider the polytope determined by a jump system, and in particular the faces of this polytope. Then, we will proceed to properties specific to two-dimensional jump systems, including a characterization. We will then consider properties for higher-dimensional jump systems. We will conclude with analysis of the constant-sum jump systems, which are equivalent to faces of jump systems.

**2. Faces and polytopes.** In this section we lay the groundwork for the remainder. We present generalizations, corollaries, and elementary proofs of some known results; we fix notation, and we give several illustrative examples.

Let  $V = \{v : v \in \{-1, 0, 1\}^S, v \neq 0\}$ , and  $V' = \{v : v \in \mathbf{R}^S, v \neq 0\}$ . If  $v \in V'$  (and, in particular, if  $v \in V$ ), we set  $\omega_v = \sup_{x \in J} v^T x$ . We call  $f_v = \{x : x \in J, v^T x = \omega_v\}$  a *face* of  $J$ . For finite jump systems, each such  $f_v$  is nonempty. Lovász has shown in [4] that  $f_v$  is, in turn, a jump system.

**Example 2.1.** Let  $J = \{(2, 3), (3, 2), (3, 3), (5, 2), (5, 3)\}$ . Then  $\omega_{(1,0)} = 5$  and  $f_{(1,0)} = \{(5, 2), (5, 3)\}$ .  $\omega_{(-1,-1)} = -5$  and  $f_{(-1,-1)} = \{(2, 3), (3, 2)\}$ .

The following useful result is a simple corollary of Theorem 1.1.

**Theorem 2.1.** Let  $v_1, v_2, \dots, v_r \in V'$  have disjoint support. Then  $f_{v_1} \cap f_{v_1+v_2} \cap \dots \cap f_{v_1+v_2+\dots+v_r} \neq \emptyset$ .

*Proof.* Let  $M = \max_{z \in J, i \in S} |z_i|$ . For  $1 \leq j \leq r$ , set  $w^j = v_1 + v_2 + \dots + v_j$ . Let  $B^j$  be defined as

$$B_i^j = \left\{ \begin{array}{ll} (-\infty, -M] & \text{if } w_i^j = -1, \\ (-\infty, \infty) & \text{if } w_i^j = 0, \\ [M, \infty) & \text{if } w_i^j = 1. \end{array} \right\}.$$

Clearly  $B^r \subseteq \dots \subseteq B^2 \subseteq B^1$ . □

Every point  $x$  of the jump system satisfies  $v^T x \leq \omega_v$  for every  $v \in V'$ . The set of all such points in  $\mathbf{Z}^S$  (not necessarily in  $J$ ) we call the *polytope associated with  $J$* , denoted  $P_J$  or  $P$ .

We call  $\{x : x \in P, v^T x = \omega_v\}$  a *face* of  $P$ . We call those points *surface* points, while the other points of  $P$  are called *interior* points. We call points in  $P \setminus J$  *gaps*.

The following two results show that  $V'$  and  $V$  induce equivalent geometry. This was first shown in [1] using bisubmodular polyhedra; we present elementary proofs. The first result, concerning faces, is actually a bit stronger, and generalizes a result in [4].

**Theorem 2.2.** *Let  $v' \in V'$ . Then  $\mathfrak{f}_{v'} \subseteq \mathfrak{f}_{\bar{v}'}$ .*

*Proof.* By reindexing and reflecting if necessary, we may assume without loss of generality that  $v'_1 \geq v'_2 \geq \dots \geq v'_{|S|} \geq 0$ . Let  $m$  be between 1 and  $|S|$  such that  $v'_m > 0$ , but  $v'_{m+1} = 0$ . Set  $v = (1, \dots, 1, 0, \dots, 0)$  where the first  $m$  coordinates are 1, and the remaining  $|S| - m$  coordinates are 0.

Suppose  $x \in \mathfrak{f}_{v'} \setminus \mathfrak{f}_v$ . Let  $y \in \mathfrak{f}_v$  be such that  $d(x, y)$  is minimal. Since  $v^T y > v^T x$ , there must exist  $i \leq m$  for which  $y_i > x_i$ . Now, since  $v'^T x \geq v'^T y$ , there must exist  $j \leq m$  for which  $x_j > y_j$ .

Consider the step  $y \xrightarrow{x} y + e_j$ . Because  $v^T(y + e_j) > v^T y = \omega_v$ , we know that  $y + e_j \notin J$ . Therefore Axiom 1.1 states that there exists a second step  $y + e_j \xrightarrow{x} y + e_j + s \in J$ . Because  $s$  is a step and  $v^T y + 1 + v^T s = v^T(y + e_j + s) \leq \omega_v = v^T y$ , we must have  $s = -e_k$  for some  $k \leq m$ . But then  $y + e_j + s \in \mathfrak{f}_v$  and  $d(y + e_j + s, x) < d(y, x)$ , which violates the minimal choice of  $y$ .  $\square$

If  $P$  is the polytope induced by  $V'$ , consider  $\bar{P}$ , the polytope analogously induced by  $V$ . It is obvious that  $P \subseteq \bar{P}$ . The following result shows that, in fact,  $\bar{P} = P$ .

**Theorem 2.3.** *Let  $x$  be a surface point of  $P$ . Then  $x$  is a surface point of  $\bar{P}$ .*

*Proof.* The result holds for  $x$  in the jump system by Theorem 2.2. Suppose it did not hold for all surface points in the polytope. Let  $x \in P$  be such a surface point. Let  $v' \in V'$  be such that  $v'^T x = \omega_{v'}$ . By reindexing and reflecting if necessary, we may assume without loss of generality that  $v'_1 \geq v'_2 \geq \dots \geq v'_{|S|} \geq 0$ . Finally, by translation, we can assume that  $x = (0, 0, \dots, 0)$ , and therefore that  $\omega_{v'} = 0$ . However, by our assumption on  $x$ , we must have  $\omega_v > 0 = v^T x$  for each  $v \in V$ .

We can now write  $v' = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_k u_k$ , where  $\lambda_i > 0$  and each  $u_i$  is a 0-1 vector of the form  $u_i = e_1 + e_2 + \dots + e_{t_i}$ , for some  $1 \leq t_1 < t_2 < \dots < t_k$ . Observe that each  $u_i \in V$ , and hence that each  $\omega_{u_i} > 0$ .

Let  $M = \max_{z \in J, i \in S} z_i$ . Consider the boxes  $B^i = [M, \infty) \times \cdots \times [M, \infty) \times \mathbf{Z}^{|S|-t_i}$ . Because  $B^1 \subseteq B^2 \subseteq \cdots \subseteq B^k$ , we can apply Theorem 1.1, which gives us some  $y$  contained in  $f_{u_1} \cap f_{u_2} \cap \cdots \cap f_{u_k}$ . Now, observe that  $v'^T y = (\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k)^T y = \lambda_1 \omega_{u_1} + \lambda_2 \omega_{u_2} + \cdots + \lambda_k \omega_{u_k} > 0 = \omega_{v'}$ . This is impossible, and therefore no such  $x$  could have existed.  $\square$

The following result shows that two “similar” points in a face force a variety of other points to be in the face as well.

**Theorem 2.4.** *Let  $J$  be a jump system, and let  $v \in V$ . Suppose  $a$  and  $b$  are in  $f_v$ . Let  $T = \{i \mid v_i(a_i - b_i) \neq 0\}$ . Suppose that  $T$  contains only two coordinates,  $\alpha$  and  $\beta$ . Then  $a' = (a_1, a_2, \dots, a_{\alpha-1}, b_\alpha, a_{\alpha+1}, \dots, a_{\beta-1}, b_\beta, a_{\beta+1}, \dots, a_{|S|})$  is in  $f_v$ , as well as every lattice point between  $a$  and  $a'$ .*

*Proof.* By reindexing, reflection, and translation, we may assume without loss of generality that for some  $m > 0$ ,  $1 = v_1 = \cdots = v_m$ ,  $0 = v_{m+1} = \cdots = v_{|S|}$ , that  $T = \{1, 2\}$ , that  $a = (0, 0, \dots, 0)$ , and that  $b = (b_1, -b_1, 0, \dots, 0, b_{m+1}, \dots, b_n)$  where  $b_1 > 0$ .

It is enough to prove that  $(1, -1, 0, \dots, 0) \in J$  because then we can recursively set  $a = (1, -1, 0, \dots, 0)$  and prove that  $(2, -2, 0, \dots, 0) \in f_v$ , and so on. Consider the step  $a \xrightarrow{b} (1, 0, \dots, 0)$ . We see that  $(1, 0, \dots, 0) \notin J$  because  $v^T(1, 0, \dots, 0) > v^T a$ . By Axiom 1.1, we can take a second step that will get us back into  $J$ . The only possible second step is  $(1, 0, \dots, 0) \xrightarrow{b} (1, -1, 0, \dots, 0)$ , because no step is possible in the direction of  $b$  between the third and  $m$ th coordinates, and a step in any of the last  $|S| - m$  coordinates will take us out of  $f_v$ . Therefore,  $(1, -1, 0, \dots, 0) \in J$ .  $\square$

**Example 2.2.** Let  $v = (1, 1, 1, 0)$ . Suppose that both  $(3, 5, 6, 7)$  and  $(0, 5, 9, 10)$  are in  $f_v$ . Then  $(0, 5, 9, 7)$ ,  $(1, 5, 8, 7)$ , and  $(2, 5, 7, 7)$  are also in  $f_v$ . Furthermore, so are  $(3, 5, 6, 10)$ ,  $(2, 5, 7, 10)$ , and  $(1, 5, 8, 10)$ .

This last corollary will be used in our characterization of two-dimensional jump systems. A more general, unpublished, result of

Sebő appears in Geelen's unpublished lecture notes [2]. This elementary result, however, is all we need in the sequel.

**Corollary 2.1.** *If  $J \subseteq \mathbf{Z}^2$  is a jump system and  $a, b \in \mathfrak{f}_{(\pm 1, \pm 1)}$ , then all the points between  $a$  and  $b$  are also in the jump system.*

**3. Two-dimensional properties and characterization.** One-dimensional jump systems are easily characterized: there can be no two adjacent gaps. Two-dimensional jump systems are more difficult. In this section we provide such a characterization, together with a theorem about gaps. In [2] it is shown that any gap must lie on a line segment between some two points of the jump system. We provide an elementary proof of this result for two-dimensional jump systems.

We start by giving the still weaker result in the special case where the gap is a surface gap. This is needed in our characterization of two-dimensional jump systems.

**Lemma 3.1.** *Let  $J \subseteq \mathbf{Z}^2$  be a jump system and  $a$  be a surface gap. Then there exist points  $x, y \in J$  (both on the same face of  $P$  as  $a$ ) such that  $a$  lies on the line segment connecting  $x$  and  $y$ .*

*Proof.* By translation we will assume without loss of generality that  $a = (0, 0)$ . By hypothesis, let  $v \in V$  be such that  $v^T a = \omega_v = 0$ . Let  $x \in \mathfrak{f}_v$ . There are two candidates  $w \in V$  such that  $|w - v| = 1$ , and a simple calculation shows that for at least one of them,  $x \notin \mathfrak{f}_w$ . Now, by Theorem 2.1, there must be some  $y \in \mathfrak{f}_v \cap \mathfrak{f}_w$ . Observe that  $x$  and  $y$  are on the same line (the face of  $P$ ), and  $a$  is between them.  $\square$

We are now ready to present a characterization of two-dimensional jump systems.

**Theorem 3.1.** *Let  $J \subseteq \mathbf{Z}^2$ . Then  $J$  is a jump system if and only if*

1. *Each face of  $J$  is a jump system, and*

2a. *For any two adjacent gaps  $x', x''$  in  $P$ , all points on the line containing them are not in  $J$ .*

(or, equivalently)

2b. *The following configuration is forbidden:  $x \in J$ ,  $x'$  and  $x''$  gaps, with  $x + 2\alpha = x' + \alpha = x''$  for some  $|\alpha| = 1$ .*

*Proof.* ( $\Rightarrow$ ). Assume that  $J$  is a jump system. The first condition is obviously true.

We will show the second condition by way of contradiction. By reindexing, reflection, and translation, we may assume that  $x = (0, 0) \in J$ ,  $x' = (1, 0) \in P \setminus J$ ,  $x'' = (2, 0) \in P \setminus J$ . This implies that there are no points in  $J$  of the form  $(k, 0)$  where  $k > 0$ .

Observe that  $x'$  cannot be on a face of  $P$ . If it were, the face would have to be  $(0, \pm 1)$  to allow  $x, x''$  to be in  $P$ , and then we get an immediate contradiction from Lemma 3.1.

We therefore assume that  $x'$  is an interior point. Since our jump system is finite, there must be some surface gap  $y = (m, 0)$  with  $m > 1$ . If  $y$  is on the  $(1, \pm 1)$  face, then we immediately get a contradiction from Lemma 3.1 and Corollary 2.1. Otherwise,  $y$  is on the  $(1, 0)$  face. We must have  $(m, 1) \in J$  by Lemma 3.1 and the one-dimensional classification. However,  $(m, 1), x$ , and  $y$  violate Axiom 1.1 since the step  $(m, 1) \xrightarrow{x} y$  will not have a second valid step toward  $x$ .

( $\Leftarrow$ ). Assume (1) and (2). We need to show Axiom 1.1 holds for any  $x, y \in J$ . By reflection and translation, we can assume that  $x = (0, 0)$  and  $y = (p, q)$  where  $p, q \geq 0$ . Furthermore, we can intersect  $J$  with a box to assume that all points in  $J$  have nonnegative coordinates. That is, we assume that  $x = f_{(-1, -1)} \cap f_{(-1, 0)} \cap f_{(0, -1)}$ .

We will first show Axiom 1.1 if  $pq = 0$ . Suppose that  $q = 0$ .  $(1, 0)$  and  $(2, 0)$  are both in  $P$ , and hence by (2) cannot both be gaps. Axiom 1.1 follows. The case  $p = 0$  follows by symmetry.

We may assume, by reindexing if necessary, that the given first step is  $x \xrightarrow{y} (1, 0)$ . Suppose that  $(1, 0) \notin J$ . We will show that either  $(2, 0)$  or  $(1, 1)$  is in  $J$ , proving Axiom 1.1.

Apply Theorem 2.1 to get  $z \in f_{(0, -1)} \cap f_{(1, -1)}$ . If  $z \neq x$ , then  $z = (m, 0)$  and therefore  $(1, 0), (2, 0)$  are both in  $P$ . Hence  $(2, 0)$  cannot be a gap by (2). On the other hand, if  $z = x$ , then  $\omega_{(1, -1)} = 0$ , and



therefore  $(1, 1) \in f_{(1,-1)}$ . By Lemma 3.1 and Corollary 2.1 we must have  $(1, 1) \in J$ .  $\square$

The following strengthens Lemma 3.1 to all gaps in two-dimensional jump systems.

**Theorem 3.2.** *Let  $J \subseteq \mathbf{Z}^2$  be a jump system and  $a$  a gap. Then there exist points  $x, y \in J$  such that  $a$  lies on the line segment connecting  $x$  and  $y$ .*

*Proof.* Without loss of generality, we assume that  $a = (0, 0)$ . We further assume that  $a$  is an interior gap, or else Lemma 3.1 would apply.

Consider  $(1, 0)$  and  $(-1, 0)$ . If both are in  $J$ , the theorem follows. If either is a gap, however, then all  $(k, 0)$  must be gaps for  $k \in \mathbf{Z}$ . Similarly we can assume that all  $(0, k)$  are gaps. By Theorem 2.1, let  $c^1 \in f_{(-1,-1)} \cap f_{(-1,0)}$ ,  $c^2 \in f_{(-1,-1)} \cap f_{(0,-1)}$ . Since  $a$  is an interior point, we must have  $c_1^1 < 0$ ,  $c_2^2 < 0$ . By Corollary 2.1, we know that all points between  $c^1$  and  $c^2$  are in  $J$ . Therefore, we can choose some  $c \in J$  with  $c_1 < 0$ ,  $c_2 < 0$ .

We will now show that  $(1, 1)$  is in  $J$ . Suppose otherwise; let  $b = (p, q) \in J$  have  $p > 0$ ,  $q > 0$  with  $p + q$  minimal. By reindexing if necessary, assume that  $p > 1$ . Consider  $b \xrightarrow{c} (p-1, q)$ . This new point must not be in  $J$  by the minimality of  $b$ . Hence, by Axiom 1.1 we must have a second step  $(p-1, q) \xrightarrow{c} d$ , with  $d \in J$ . But then either  $d$  violates the minimality of  $b$ , or else has a coordinate equal to zero, which is also forbidden. This contradiction shows that  $(1, 1)$  is in  $J$ .

By a symmetric argument, we must have  $(-1, -1)$  in  $J$ . And now the theorem follows.  $\square$

**4. Geometry of higher-dimensional jump systems.** In this section we include several additional geometric results. The configuration of Theorem 3.1, while no longer forbidden, imposes a variety of restrictions on  $J$ , particularly for three-dimensional jump systems. We also include another forbidden configuration (that, unfortunately, does not characterize higher-dimensional jump systems). But first, we have

the following result, that each hyperquadrant relative to a gap must contain some point of  $J$ .

**Theorem 4.1.** *Let  $x$  be a gap. Let  $v \in \{-1, 1\}^{|S|}$ . Then  $\{y \in J : v_i(y_i - x_i) \geq 0\}$  is nonempty.*

*Proof.* By translation, we can assume without loss of generality that  $x$  is the origin. By reflection, we may assume without loss of generality that  $v = (1, 1, \dots, 1)$ . For convenience, for each  $T \subseteq \{1, 2, \dots, |S|\}$ , we define the set  $N_T = \{y \in J : \sum_{i \in T} y_i \geq \sum_{i \in T} x_i\}$ . The theorem follows if we can show that  $N_1 \cap N_2 \cap \dots \cap N_{|S|}$  is nonempty.

We will show this in  $|S|$  steps. Each step will allow for any permutation  $i_1, i_2, \dots, i_{|S|}$  of  $1, 2, \dots, |S|$ . The first step is to show that  $N_{i_1} \cap N_{i_1 i_2} \cap \dots \cap N_{i_1 i_2 \dots i_{|S|}}$  is nonempty. This follows immediately from Theorem 2.1, as  $f_{e_{i_1}} \cap f_{e_{i_1} + e_{i_2}} \cap \dots \cap f_{e_{i_1} + e_{i_2} + \dots + e_{i_{|S|}}} \subseteq N_{i_1} \cap N_{i_1 i_2} \cap \dots \cap N_{i_1 i_2 \dots i_{|S|}}$ . We say that this step admits one coordinate, as there is one term  $N_{i_1}$  with just one coordinate.

We now assume that we have completed step  $k$ , for  $1 \leq k \leq |S| - 1$ . That step admits  $k$  coordinates:  $N_{i_1} \cap N_{i_2} \cap \dots \cap N_{i_k} \cap N_{i_1 i_2 \dots i_k i_{k+1}} \cap N_{i_1 i_2 \dots i_k i_{k+1} i_{k+2}} \cap \dots \cap N_{i_1 i_2 \dots i_{|S|}} \neq \emptyset$ . It suffices to show that we can admit  $k + 1$  coordinates:  $N_{i_1} \cap N_{i_2} \cap \dots \cap N_{i_k} \cap N_{i_{k+1}} \cap N_{i_1 i_2 \dots i_k i_{k+1} i_{k+2}} \cap \dots \cap N_{i_1 i_2 \dots i_{|S|}} \neq \emptyset$ .

Choose  $z \in N_{i_1} \cap N_{i_2} \cap \dots \cap N_{i_k} \cap N_{i_1 i_2 \dots i_k i_{k+1}} \cap \dots \cap N_{i_1 i_2 \dots i_{|S|}}$  with  $z_{i_{k+1}}$  maximal. If  $z_{i_{k+1}} \geq 0$  this  $(k + 1)^{\text{th}}$  step is complete, so assume otherwise. Because  $z \in N_{i_1 i_2 \dots i_k i_{k+1}}$ , we have  $z_{i_1} + z_{i_2} + \dots + z_{i_{k+1}} \geq 0$ . But  $z_{i_{k+1}} < 0$ , so for some other coordinate (say  $i_1$ ),  $z_{i_1} > 0$ . Now, choose  $y \in N_{i_2} \cap N_{i_3} \cap \dots \cap N_{i_{k+1}} \cap N_{i_1 i_2 \dots i_k i_{k+1}} \cap \dots \cap N_{i_1 i_2 \dots i_{|S|}}$ . Because  $z_{i_{k+1}} < 0 \leq y_{i_{k+1}}$ , we have  $z \xrightarrow{y} z + e_{i_{k+1}}$ . By the maximal choice of  $z$ , we must have  $z + e_{i_{k+1}} \notin J$ . So, by Axiom 1.1, we must have  $z + e_{i_{k+1}} \xrightarrow{y} z + e_{i_{k+1}} + \alpha$ . But, again by the maximal choice of  $z$ , we must have  $z + e_{i_{k+1}} + \alpha \notin N_{i_1} \cap N_{i_2} \cap \dots \cap N_{i_k} \cap N_{i_1 i_2 \dots i_k i_{k+1}} \cap \dots \cap N_{i_1 i_2 \dots i_{|S|}}$ . Since this last step was in the direction of  $y$ , we must have  $z + e_{i_{k+1}} + \alpha \notin N_{i_1}$ . That is,  $\alpha = -e_{i_1}$  and  $z_{i_1} = 0$ . But this is a contradiction since  $z_{i_1} > 0$ .

□

If the dimension of  $J$  is greater than two, the configuration of Theorem 3.1 is no longer forbidden. However, it does impose some conditions on  $J$ , as the following two results demonstrate. The first shows that the configuration prohibits a variety of points from being in  $J$ .

**Theorem 4.2.** *Let  $x \in J$ ,  $x'$  and  $x''$  gaps, and  $x + 2\alpha = x' + \alpha = x''$  for some  $|\alpha| = 1$ . For any  $y \in J$ , decompose  $y$  as  $y = x + k_y\alpha + \hat{y}$ , for  $\hat{y} \cdot \alpha = 0$ . Then, we must have  $|\hat{y}| \geq k_y$ .*

*Proof.* Suppose otherwise. Choose some  $y = x + k_y\alpha + \hat{y}$  with  $|\hat{y}| < k_y$  and  $k_y$  minimal.  $\hat{y}$  cannot be 0, since then  $y \xrightarrow{x} y - \alpha$  would violate Axiom 1.1. Therefore, there must be some step  $y'$  from  $y$  toward  $x$  in some coordinate not corresponding to  $\alpha$ . That is,  $y \xrightarrow{x} y'$ . But we have  $|\hat{y}'| < |\hat{y}| < k_y = k_{y'}$ . By the minimal choice of  $y$ , we must have  $y' \notin J$ . Now, by applying Axiom 1.1, we get  $y'' \in J$ , with  $y' \xrightarrow{x} y''$ . By the minimal choice of  $y$ , we must have  $|\hat{y}''| \geq k_{y''}$ . However, this is a contradiction, since either  $k_{y''} = k_y - 1 > |\hat{y}| - 1 = |\hat{y}''|$  or  $k_{y''} = k_y > |\hat{y}| \geq |\hat{y}''|$ .  $\square$

For three-dimensional jump systems, this configuration actually forces quite a bit more.

**Theorem 4.3.** *Let  $J \subseteq \mathbf{Z}^3$ . Let  $x \in J$ ,  $x'$  and  $x''$  gaps, and  $x + 2e_3 = x' + e_3 = x''$ . Then the eight points in  $\{x'' \pm e_1 \pm e_2; x' \pm e_1; x' \pm e_2\}$  are all in  $J$ .*

*Proof.* By translation, we may assume without loss of generality that  $x$  is the origin. By reflection, the theorem will follow if we can show that the three points  $x'' + e_1 + e_2, x' + e_1, x' + e_2$  are all in  $J$ . By Theorem 4.1, there must be some  $a \in J$  with  $a_1 \geq 0, a_2 \geq 0, a_3 \geq 2$ . Now, consider  $x \xrightarrow{a} x'$ . By Axiom 1.1, there must be a second step  $x' \xrightarrow{a} b$ . We must have  $b \in J$ , with  $b = x' + e_1$  or  $b = x' + e_2$ . By reindexing if necessary, we may assume without loss of generality that  $b = x' + e_1$ .

By Theorem 4.1, there must be some  $c \in J$  with  $c_1 \leq 0, c_2 \geq 0, c_3 \geq 2$ . Consider  $b \xrightarrow{c} x'' + e_1$ . This is not in  $J$  by Theorem 4.2. Thus, by Axiom 1.1, there must be a second step  $x'' + e_1 \xrightarrow{a} d$ . We must have  $d \in J$ , with  $d = x'' + e_1 + e_3, d = x''$ , or  $d = x'' + e_1 + e_2$ . The first is impossible by Theorem 4.2, and the second is impossible by the hypotheses. Hence we must have  $d = x'' + e_1 + e_2 \in J$ .

Finally, consider  $d \xrightarrow{x} x'' + e_2$ . By Theorem 4.2, this is not in  $J$ . Thus, by Axiom 1.1, there must be a second step  $x'' + e_2 \xrightarrow{x} e$ . We must therefore have  $e \in J$ , with  $e = x' + e_2$ .  $\square$

Our final result of this section concerns rifts. For  $v \in V$  and  $b \in \mathbf{Z}$ , the set of points  $\{x : v^T x = b\}$  is called a *rift*  $R(v, b)$  whenever none of those points is in  $J$ . We say that  $J$  *admits*  $R(v, b)$ . The result states that if  $J$  admits two adjacent rifts, it must be entirely on one side or the other of the rifts.

**Theorem 4.4.** *Suppose  $J$  admits both  $R(v, b)$  and  $R(v, b+1)$ . Then, either  $\omega_v < b$ , or  $\omega_{-v} < -(b+1)$ .*

*Proof.* Let  $x, y \in J$  be such that  $v^T x < b, v^T y > b+1$  is chosen so that  $|x - y|$  is minimal. Consider any step  $x \xrightarrow{y} z$ . If  $v^T z = b$ , then  $z$  is in the rift  $R(v, b)$  and hence not in  $J$ . If  $v^T z < b$ , then by the minimal choice of  $x$  we again have that  $z \notin J$ . Hence, by Axiom 1.1, there is some step  $z \xrightarrow{y} w$  with  $w \in J$ . Since  $v^T z \leq b, v^T w \leq b+1$ . But we must therefore have  $v^T w < b$ , because of the two rifts. And now,  $w, y \in J$  violate the minimality of  $x, y$ .  $\square$

**5. Constant-sum jump systems.** We now turn our attention to the special case where, for some  $v \in V$ , we have  $J = f_v$ . The result of this section is a characterization of these constant-sum jump systems in terms of an operation we call strong reduction.

Let  $J$  be a collection of points, let  $T \subseteq S$  with  $|T| \geq 2$ , and let  $\alpha \in \{-1, 0, 1\}^T$ . Then the *strong reduction*  $J[\alpha \cdot T]$  is defined by  $J[\alpha \cdot T] = \{x' \in \mathbf{Z}^{(S \setminus T) \cup 0} : x \in J, x'_0 = \sum_{i \in T} \alpha_i x_i, x'_j = x_j \text{ (for } j \notin T)\}$ . Observe that the operation is equivalent to reflection followed by projection followed by reduction.

**Example 5.1.** Let  $J = \{(1, 1, 1), (2, 1, 1), (1, 2, 1), (2, 2, 1), (1, 1, 2), (2, 2, 2)\}$ . This is a jump system.  $J[x_1 + 0x_2] = \{(1, 1), (2, 1), (1, 2), (2, 2)\}$ ,  $J[x_1 + x_2] = \{(2, 1), (3, 1), (4, 1), (2, 2), (4, 2)\}$ ,  $J[x_1 - x_2] = \{(-1, 1), (0, 1), (1, 1), (0, 2)\}$ ,  $J[0x_1 - x_2 - x_3] = \{(-4), (-3), (-2)\}$ .

Because all of its constituent operations preserve Axiom 1.1, strong reduction does as well. Our final result is a partial converse to this fact, restricted to constant-sum jump systems. Unfortunately, it cannot be generalized to arbitrary jump systems, as  $J = \{(0, 0), (2, 0), (2, 2)\}$  is a collection of points that does not satisfy Axiom 1.1, but every strong reduction of which does.

**Theorem 5.1.** *Let  $J$  be a collection of points, with  $v \in V$  such that  $v^T x$  is constant for all  $x \in J$ . Then  $J$  is a jump system if and only if every strong reduction of  $J$  is a jump system.*

*Proof.* Suppose that, in violation of Axiom 1.1, there are points  $a, b \in J$  and a step  $a \xrightarrow{b} s$  with  $s \notin J$ , and no second step from  $s$  to  $b$  in  $J$ . By translation, we may assume that  $a$  is the origin (and hence the origin is in any strong reduction of  $J$ ), and hence that  $\omega_v = 0$ . By reindexing, we may assume that  $s = \pm e_1 = \bar{b}_1$ . By reindexing the remaining coordinates, we may assume that all coordinates after the  $m$ th are zero, for some  $1 \leq m \leq |S|$ . Finally, by reflection, we may assume that either  $v = (1, 1, \dots, 1, 0, \dots, 0)$ , or  $v = (0, 1, \dots, 1, 0, \dots, 0)$ . These two cases will be treated separately. In both, consider  $J' = J[x_1 + 0x_2]$ , a jump system by hypothesis. Set  $e = (\bar{b}_1, 0, \dots, 0)$ . This is a step from the origin toward  $(b_1, b_3, \dots, b_n) \in J'$ . In both cases, we will show that  $e \notin J'$ , and hence by Axiom 1.1 we must have  $e \xrightarrow{(b_1, b_3, \dots, b_n)} f$ , with  $f = e + \bar{b}_j \in J'$  for some  $1 \leq j \leq |S|$ . We will then show that this leads to a contradiction, for any possible  $j$ .

*Case 1.* Since  $v^T b = 0$ , there must be some coordinate  $k$  between 2 and  $m$  with  $\bar{b}_k = -\bar{b}_1$ . Without loss of generality, we may reindex and assume that  $k = 2$ . Now, if  $e \in J'$ , then  $(\bar{b}_1, \alpha, 0, \dots, 0) \in J$  for some  $\alpha$ , hence by the constant-sum property we have  $\alpha = -\bar{b}_1$ . But then this is a step in  $J$  from  $s$  toward  $b$ , which by assumption we cannot have. Hence  $e \notin J'$ , and so we get  $f = e + \bar{b}_j \in J'$ .

First, we will consider the cases where  $j = 1$  or  $j > m$ . By the constant-sum property, this implies that  $c = (2\bar{b}_1, 2\bar{b}_2, 0, \dots, 0) \in J$  (respectively, that  $c = (\bar{b}_1, \bar{b}_2, 0, \dots, \bar{b}_j, \dots, 0) \in J$ ). Consider  $J'' = J[x_1 - x_2]$ , a jump system by hypothesis. If  $e \in J''$ , then  $(\alpha, \beta, 0, \dots, 0) \in J$  for some  $\alpha - \beta = \bar{b}_1$ . By the constant-sum property, we also know that  $\alpha + \beta = 0$ . These two equations cannot be satisfied by integers, and therefore  $e \notin J''$ . If  $2e \in J''$ , then  $(\alpha, \beta, 0, \dots, 0) \in J$  for some  $\alpha - \beta = 2\bar{b}_1$ . By the constant-sum property, we also know that  $\alpha + \beta = 0$ . There is one solution to these equations—that  $(\bar{b}_1, -\bar{b}_1, 0, \dots, 0) \in J$ ; however, this is a step from  $s$  toward  $b$ , which by assumption is disallowed. Hence  $2e \notin J''$ . Finally, if  $j > m$  and  $f \in J''$ , then  $(\alpha, \beta, \dots, 0, \bar{b}_j, 0, \dots, 0) \in J$ , for some  $\alpha - \beta = \bar{b}_1$ . By the constant-sum property, we also have that  $\alpha + \beta = 0$ . These two equations cannot be satisfied by integers, and therefore  $f \notin J''$ . Now, because  $c \in J$ , we have  $c' = (4\bar{b}_1, 0, \dots, 0) \in J''$  (respectively,  $c' = (2\bar{b}_1, 0, \dots, \bar{b}_j, \dots, 0) \in J''$ ). So, we have a step  $0 \xrightarrow{c'} e$ , with  $e \notin J''$ . By Axiom 1.1, we must have a second step from  $e$  toward  $c'$  in  $J''$ . However, the only possibilities are  $2e$  and  $f$ , and we have shown that neither can be in  $J''$ .

Now, we will consider the cases where  $3 \leq j \leq m$ . Without loss of generality, we will reindex and assume that  $j = 3$ . By the constant-sum property, and our assumption, we must have that  $c = (\bar{b}_1, 2\bar{b}_2, \bar{b}_1, 0, \dots, 0) \in J$ . Consider  $J''' = J[x_3 - x_2]$ , a jump system by hypothesis. Set  $e' = (0, \bar{b}_1, 0, \dots, 0)$ ,  $e'' = (\bar{b}_1, \bar{b}_1, 0, \dots, 0)$ . If  $e' \in J'''$ , then  $(\bar{b}_1, \alpha, \beta, 0, \dots, 0) \in J$ , for some  $\beta - \alpha = 0$ . By the constant-sum property, we also know that  $\bar{b}_1 + \alpha + \beta = 0$ . These two equations cannot be satisfied by integers, and therefore  $e' \notin J'''$ . If  $e'' \in J$ , then  $(\bar{b}_1, \alpha, \beta, 0, \dots, 0) \in J$ , for some  $\beta - \alpha = \bar{b}_1$ . By the constant-sum property, we also know that  $\bar{b}_1 + \alpha + \beta = 0$ . There is one solution to these equations—that  $(\bar{b}_1, -\bar{b}_1, 0, \dots, 0) \in J$ ; however, this is a step from  $s$  toward  $b$ , which by assumption is disallowed. Hence  $e'' \notin J'''$ . Now, because  $c \in J$ , we have  $c' = (3\bar{b}_1, \bar{b}_1, 0, \dots, 0) \in J'''$ . So, we have a step  $0 \xrightarrow{c'} e'$ , with  $e' \notin J'''$ . By Axiom 1.1, we must have some second step in  $J'''$  from  $e'$  toward  $c$ . But the only possible second step is  $e''$ , which we have shown cannot be in  $J'''$ .

*Case 2.* If  $e \in J'$ , then by the constant sum property,  $(\bar{b}_1, 0, \dots, 0) = s \in J$ , which violates the hypothesis. So  $e \notin J'$ , and we get  $f = e + \bar{b}_j \in J'$ . If  $j = 1$  or  $j > m$ , then  $(2\bar{b}_1, 0, \dots, 0) \in J$  (respectively,  $(\bar{b}_1, 0, \dots, \bar{b}_j, \dots, 0) \in J$ ), which is a step from  $s$  toward  $b$ , violating the hypothesis. By reindexing, we now assume without loss of generality that  $j = 3 \leq m$ . Since  $f \in J'$ , we must have  $c = (\bar{b}_1, -\bar{b}_3, \bar{b}_3, 0, \dots, 0) \in J$ .

We now consider  $J'' = J[x_3 - x_2]$ , a jump system by hypothesis. Since  $c \in J$ , we must have  $c' = (2\bar{b}_3, \bar{b}_1, 0, \dots, 0) \in J''$ . Consider  $(0, \bar{b}_1, 0, \dots, 0)$ . This is not in  $J''$  since  $s \notin J$ . However, it is a step from the origin toward  $c'$ . Hence, by Axiom 1.1, we must have a second step toward  $c'$ . This can only be  $(\bar{b}_3, \bar{b}_1, 0, \dots, 0) \in J''$ . But then  $(\bar{b}_1, \alpha, \beta, 0, \dots, 0) \in J$ , for some  $\beta - \alpha = \bar{b}_3$ . But, by the constant-sum property, we have  $\beta + \alpha = 0$ . These two equations cannot be satisfied by integers, and hence  $J''$  cannot be a jump system, in violation of the hypothesis.  $\square$

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