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Local Stability Implies Global Stability for the Planar Ricker Competition Model

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Local stability implies global stability for the planar Ricker competition model

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Abstract

Under certain analytic and geometric assumptions we show that local stability of the coexistence (positive) fixed point of the planar Ricker competition model implies global stability with respect to the interior of the positive quadrant. This result is a confluence of ideas from Dynamical Systems, Geometry, and Topology that provides a framework to the study of global stability for other planar competition models.

Keys Words: Competition models, Local stability, Global stability, Critical curves, Compact invariant set, Principal preimage function, fold, cusp.

1 Introduction

The question of global asymptotic stability has been of great interest in both differential and difference equations for over five decades. Markus and Yamabe [18] conjectured that the origin is globally asymptotically stable in the differential equation $X'(t) = F(X(t))$, where F is defined on $\mathbb{R}^n$ if the origin is the unique fixed point of F and the Jacobian matrix of F has negative real parts at every point in

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$\mathbb{R}^n$. Later, Fessler [8], Glutsyuk [10], and Gutierrez [11] independently showed that the conjecture is true in the plane, that is, $\mathbb{R}^2$.

LaSalle stated the discrete analog of the Markus-Yamabe Conjecture, namely if the spectral radius of the Jacobian Matrix $JF(X)$ of a map F on $\mathbb{R}^n$ at every point in $\mathbb{R}^n$ is less than 1, then the unique fixed point at the origin is globally asymptotically stable. Chamberland [2] and Martelli [19] showed, independently, that LaSalle conjecture is false even for planar maps. While this approach to global stability may be of great interest in differential equations and continuous dynamical systems, it does not, however, receive the same attention in difference equations and discrete dynamical systems. This is due to the fact that the above conjectures assume severe conditions on the spectral radius of the Jacobian of the maps that are not satisfied even for the most studied one-dimensional unimodal maps.

In our view, the focus of research on global stability should lie on the question of when does local stability implies global stability. This approach has been very fruitful in the case of one-dimensional unimodal maps. It is well-known (see, Devaney [5], Elaydi [6], and Liz [16]) that in many well-know unimodal maps, such as the logistic and the Ricker maps, local asymptotic stability of the fixed point implies global asymptotic stability. Moreover, in Cull [4] and Sharkovsky et al [23], conditions were given under which local asymptotic stability would imply global asymptotic stability of the unique fixed point of a unimodal map.

This approach has been adopted by Hal Smith [24] where he showed that local asymptotic stability implies global asymptotic stability for two-dimensional monotone maps. This is a complete departure from LaSalle conjecture and while the latter assumption is much weaker than that of LaSalle conjecture, the assumption of monotonicity is rather restrictive.

In this paper we will address the global stability question in the spirit of Smith's results but without the restriction of monotonicity. Our focus will be on the Ricker competition map, however we do expect our results to be extended to other classes of non-invertible planar maps, such as the logistic map and the Cournout duoploy competition model.

Our approach utilizes a set of tools from several areas of mathematics. Our first tool is the notion of critical curves (general folds), originally introduced by Whitney [25] and later popularized by Mira and his collaborators [1, 9, 15, 20, 21, 22]. The second tool we use is the topological notion of exposed points. We will use this notion to describe the geometry of the image of our maps and detect sets where the map is injective. Finally, we will use the notion of slow and fast stable manifold and the notion of global unstable manifold to complete the proof of our results.

The results in this paper are related to the work done in [12] and [17], where the authors gave necessary and sufficient conditions for the local asymptotic stability of the positive fixed point of both the logistic and Ricker competition models, respectively. It was shown that if the significant parameters, namely the carrying capacities of the two species, lie in a certain region, called the stability region, then the positive equilibrium point is locally asymptotically stable. Since the map has four parameters, the task of determining the stability region in the parameter space is a formidable task, see Figure 1.

In Elaydi and Luís [7], it was conjectured that if the positive equilibrium of the Ricker competition model and other competition models as well is locally asymptotically stable, then it must be globally asymptotically stable with respect to the interior of the first quadrant. In this paper we prove this conjecture under some analytic and geometric conditions for the Ricker competition model. Our novel approach reveals a confluence of ideas from Dynamical Systems, Geometry, and Topology that can be used to study global stability for other planar competition models. For now, we will focus in developing and establishing this theory for the Ricker competition map.

2 Preliminaries

We believe that it is worthwhile to review what have been done in [17] in regards to the question of the local stability of the positive fixed point of the Ricker competition map. This previous work is the first complete determination of the stability region in the parameter space that produced the bifurcation diagram of Figure 1. Note that Figure 1 not only provide us with the stability regions of all the three fixed points of the map, but more importantly it shows that there is a period-doubling bifurcation scenario reminiscent of the dynamics of one-dimensional unimodal maps. It also shows that the possible presence of bubbles in the bifurcation diagrams of each species separately.

We begin by stating the following planar Ricker type model with population numbers as state variable

$$\begin{cases} p_{n+1} = p_n \exp(r - c_{11}p_n - c_{12}q_n) \\ q_{n+1} = q_n \exp(s - c_{21}p_n - c_{22}q_n) \end{cases}, \tag{2.1}$$

where the parameters r and s are the inherent exponential growth rates at low densities and $c_{i,j}$, $i, j = 1, 2$, are the competition intensity coefficients measuring the effects of intra-specific competition and inter-specific competition, with units 1/(population units). More precisely, c_{11} and c_{22} are the intra-specific competition parameters while c_{12} and c_{21} are the inter-specific competition parameters. Notice that, these six parameters are assume to be positive.

Scaling the state variables against the inherent carrying capacities

$$u = \frac{c_{11}}{r}p \text{ and } v = \frac{c_{22}}{s}q$$

we obtain

$$\begin{cases} u_{n+1} = u_n \exp(r - ru_n - \frac{c_{12}}{c_{22}} sv_n) \\ v_{n+1} = v_n \exp(s - \frac{c_{21}}{c_{11}} ru_n - sv_n) \end{cases}.$$

Finally, setting

$$x = ru \text{ and } y = sv$$

we get

$$\begin{cases} x_{n+1} = x_n \exp(r - x_n - ay_n) \\ y_{n+1} = y_n \exp(s - bx_n - y_n) \end{cases}, \tag{2.2}$$

where we denote $a = \frac{c_{12}}{c_{22}}$ and $b = \frac{c_{21}}{c_{11}}$, for simplicity. We remark here that we call System (2.2) by Ricker competition model.

Let us represent System (2.2) by the following map

$$F(x, y) = (f_1(x, y), f_2(x, y)) = (xe^{r-x-ay}, ye^{s-y-bx}). \tag{2.3}$$

The map F may possess a coexistence (positive) equilibrium point $X^* = (x^*, y^*)$ given by

$$(x^*, y^*) = \left(\frac{as - r}{ab - 1}, \frac{br - s}{ab - 1} \right).$$

The coexistence equilibrium point exists if and only if

$$as < r \text{ and } br < s \tag{2.4}$$

or

$$as > r \text{ and } br > s. \tag{2.5}$$

Notice that (2.4) implies that $ab < 1$ while (2.5) implies $ab > 1$.

When $ab > 1$, the coexistence equilibrium point is a saddle and the asymptotic attractor of an orbit of (2.2) depends on its initial conditions. If $ab = 1$, the system (2.2) has no coexistence equilibrium point. Henceforth, we shall assume that $ab < 1$. We shall comment that this condition means that the inter-specific competition is less than the intra-specific competition since $c_{12}c_{21} < c_{11}c_{22}$.

An important result in [17] establishes criteria for the local stability of the coexistence equilibrium point (x^*, y^*) of the Ricker competition model. Namely,

Theorem 2.1 ([17])**.** *Let $a, b > 0$ such that $ab < 1$. The coexistence fixed point*

$$(x^*, y^*) = \left(\frac{as - r}{ab - 1}, \frac{br - s}{ab - 1} \right)$$

of the Ricker competition model (2.2) is locally asymptotically stable if and only if

$$4(ab - 1) + 2(1 - a)s + 2(1 - b)r \leq (as - r)(br - s) < (1 - a)s + (1 - b)r. \tag{2.6}$$

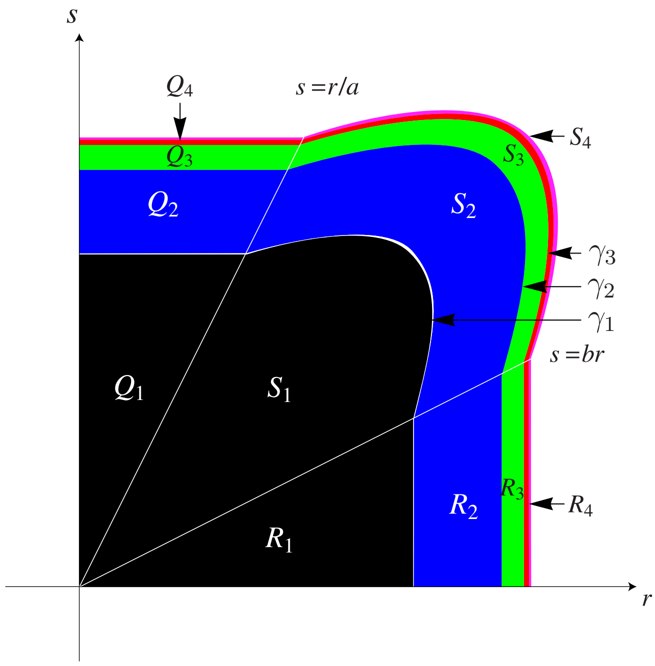

Figure 1: The stability regions and the bifurcation scenario of the Ricker competition model (2.2), in the parameter space (r, s), when the competition parameters a and b are fixed such that $ab < 1$. S_1 is the stability region of the coexistence equilibrium point (x^*, y^*), R_1 is the stability region of the exclusion fixed point $(r, 0)$, and Q_1 is the stability region of the exclusion fixed point $(0, s)$. A period-doubling bifurcation scenario occurs (in the coexistence case) as we cross from S_1 to S_2, to S_3, etc. Similarly, one has a period-doubling scenario (in the exclusion case) in the $x-$axis as we cross from R_1 to R_2, to R_3, etc. Similarly, for Q_i, $i = 1, 2, \ldots$

The stability region in the parameter space (r, s) is denoted by S_1 and is shown in Figure 1. More precisely, S_1 is the region in the $r - s$ plane bounded by the lines $s = r/a$, $s = br$ and the curve γ_1, where γ_1 is part of a branch of the hyperbola defined by

$$br^2 + 2(1-b)r - (1+ab)rs + 2(1-a)s + as^2 + 4(ab-1) = 0.$$

Equivalently, in the region defined by (2.6) and under assumption (2.4), the coexistence fixed point is locally asymptotically stable if and only if $(r, s) \in int(S_1) \cup \gamma_1$, where $int(S_1)$ denotes the interior of the region S_1.

Our main result in this paper establishes that in the region S_1 the coexistence fixed point of the Ricker competition model is globally asymptotically stable with respect to the interior of the positive quadrant. We are able to show this fact when the parameters r and s are between 1 and 2 and under some geometric conditions that will be explain in details in Section 6

Before end this section, we should mention that Smith [24] used monotonicity to prove the global stability of the fixed point of the system

$$\begin{cases} u_{n+1} = u_n \exp(r(1 - u_n - Bv_n)) \\ v_{n+1} = v_n \exp(s(1 - Cu_n - v_n)) \end{cases}, \tag{2.7}$$

when $r, s \leq 1$, in which the invariant set is $[0, \frac{1}{r}] \times [0, \frac{1}{s}]$. Notice that by the change of variables $x = ru$ and $y = sv$, System (2.7) is equivalent to

$$\begin{cases} x_{n+1} = x_n \exp(r - x_n - \frac{Br}{s} y_n) \\ y_{n+1} = y_n \exp(s - y_n - \frac{Cs}{r} x_n) \end{cases}. \tag{2.8}$$

Rewriting the parameters $a = \frac{Br}{s}$ and $b = \frac{Cs}{r}$, it follows that System (2.8) is equivalent to System (2.2). Observe that condition $ab < 1$ is equivalent to $BC < 1$.

Under the assumptions that r and s are in the unit interval, System (2.7) is monotone. So Smith's result states that when both of the carrying capacities r and s are in the unit interval, the local stability of the coexistence fixed point of (2.2) implies global stability with respect to the interior of the first quadrant.

3 Critical curves and the singularities

In this section, we introduce concepts from singularity theory and topology that will be used in our results. Let us first review some nomenclature present in the classical work of Whitney [25].

Throughout this section, let F be a differentiable map defined on a open set $U \subseteq \mathbb{R}^2$. For a point $p \in U$, we will denote the Jacobian matrix of the map F at p by $JF(p)$ and the determinant of the Jacobian matrix by $\det JF(p)$. In addition, whenever it is clear that we are referring to $\det JF(p)$, we will use $J(p)$.

The map F is said to be *regular* at a point $p = (x, y)$ if $J(p) \neq 0$. Otherwise, we say that F is *singular* at p. One of our approaches in studying the map F is to understand its image by considering its regular and singular set.

Definition 3.1. Let F be a C^2 map defined on an open subset $U \subseteq \mathbb{R}^2$. We say that $p \in U$ is a *good* point if either $J(p) \neq 0$ or $\nabla J(p) \neq 0$, where ∇ denotes the gradient. We say that the map F is *good* if every point of U is good.

Observe that $\nabla J(p) \neq 0$ if either $J_x(p)$ or $J_y(p)$ is not zero. The following lemma justifies the usage of the phrase *critical curves*.

Lemma 3.2. *Let F be a good map defined on $U \subseteq \mathbb{R}^2$. Then the singular points of the map F form differentiable curves in U, called the critical curves of F.*

The proof of this lemma can be found in [25, p. 378] and although it is simple, it is worthwhile to repeat it here. Indeed, let p be a singular point of the good map F. Then $J(p) = 0$ and $\nabla J(p) \neq 0$. Hence by the implicit function theorem, the solutions of $J(p) = 0$ lie on smooth curves.

We observe that Mira [1] defined the fundamental critical curve of a $2-$dimensional continuous good map F as the set of points for which the Jacobian determinant of F vanishes, or for which the map F is not differentiable. In other words,

$$LC_{-1} = \{p \in U : J(p) = 0, \text{ or } F \text{ is not differentiable in } p\}.$$

Let $\phi(t)$ be a C^2 parametrization of the critical curve LC_{-1} through p, with $\phi(0) = p$. We say p is a *fold point* of F if

$$\frac{d}{dt}(F \circ \phi)(0) \neq 0, \tag{3.1}$$

and p is a *cusp point* of F if

$$\frac{d}{dt}(F \circ \phi)(0) = 0 \text{ and } \frac{d^2}{dt^2}(F \circ \phi)(0) \neq 0. \tag{3.2}$$

Note that p is a fold point of F if the curve $LC_0 = F(LC_{-1})$, that is, the image of LC_{-1} is a smooth curve with nonzero tangent vector at $F(p)$, and p is a cusp point if the tangent vector is zero at $F(p)$ but becomes nonzero at a positive rate as we move away from p on LC_{-1}. It should be noted that it follows from the definition that cusp points are isolated.

Definition 3.3. A point p is an *excellent* point of a good map F if it is either a regular, fold, or a cusp point. The map F is excellent if each point of its domain is excellent.

We will later show in Section 4 that the planar Ricker competition model has exactly one cusp point on the critical curve LC_{-1} and all other points are folds. In other words, we will show that the Ricker map is excellent. This is very important because the structure of the map near such points is known and has been characterized. In the case of fold points we have the following result.

Theorem 3.4 (Theorem 15A, [25])**.** *Let $F : U \to \mathbb{R}^2$ be a differentiable map. If $p \in U$ is a fold point, then there are smooth coordinates (x_1, y_1) and (x_2, y_2) around p and $F(p)$ such that F takes the form $x_2 = x_1$ and $y_2 = y_1^2$.*

The next theorem deals with the structure at cusp points.

Theorem 3.5 (Theorem 16A, [25])**.** *Let $F : U \to \mathbb{R}^2$ be a smooth map. If $p \in U$ is a cusp point, then there are smooth coordinates (x_1, y_1) and (x_2, y_2) around p and $F(p)$ such that F takes the form $x_2 = x_1$ and $y_2 = y_1^3 - x_1 y_1$.*

The coordinate systems introduced in Theorem 3.4 and 3.5 are called the *normal forms* for a fold and a cusp, respectively. The structure of F, in a normal form, at a fold and at a cusp is depicted in Figure 2.

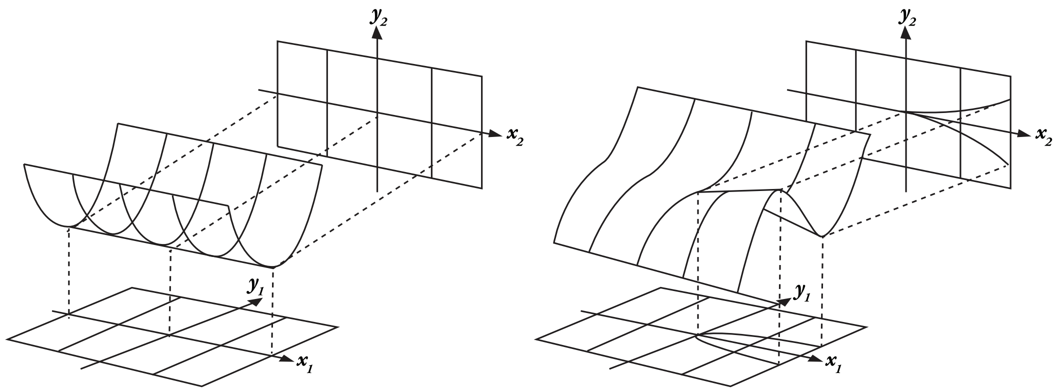

Figure 2: On the left, a depiction of the local structure near a fold point and on the right near a cusp point under a normal coordinate form.

We conclude this section with some ideas and results from differential topology to study global injectivity. In this paper, we use these ideas to develop our geometric analysis of the Ricker map as well as to ensure global injectivity in closed regions of the domain.

Definition 3.6. Let $U \subseteq \mathbb{R}^2$ be a compact region, $p \in U$, and $v \in S^1$, that is, a point in the unit circle. We say that p is *exposed in the direction of* v if there exists $\varepsilon > 0$ such that the ray $r_v(t) = p + tv \in U$ for $t \in (0, \varepsilon)$.

We remark that if $p \in int(U)$, then p is exposed in every direction. In our applications, the notion of exposed points will be used to find other points belonging to the region U. Namely, if $p \in U$ is exposed in the direction of v, then for some $t > 0$, $r_v(t) \in \partial U$. Geometrically, this is to say that the ray in the direction v eventually will hit the boundary of U.

In order to use this concept to establish global injectivity of maps in certain compact regions we will need the following very interesting result.

Theorem 3.7 (Kestelman [13])**.** *Let $F : K \to \mathbb{R}^n$ be an open and locally injective map. If $K \subseteq \mathbb{R}^n$ is a compact set, ∂K is connected, and $F|_{\partial K}$ is injective, then F is injective on K.*

Simply said, Theorem 3.7 states that in order to show injectivity on a compact region, it suffices to show injectivity in the boundary if the map is locally invertible.

4 The geometry of the Ricker Map

Our main focus will be on the Ricker competition map (2.2). Let us recall its definition, $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ given by

$$F(x, y) = (f_1(x, y), f_2(x, y)) = (xe^{r-x-ay}, ye^{s-y-bx})$$

where $r, s, a, b > 0$. The Jacobian matrix of F is given by

$$JF(x, y) = \begin{bmatrix} (1-x)e^{r-x-ay} & -axe^{r-x-ay} \\ -bye^{s-y-bx} & (1-y)e^{s-y-bx} \end{bmatrix}, \tag{4.1}$$

and consequently the determinant is given by

$$J(x, y) = -e^{r+s-x-bx-y-ay}(-1 + x + y - xy + abxy). \tag{4.2}$$

Hence if follows that $\det JF(x, y) = 0$ if and only if

$$y = \frac{1-x}{1-(1-ab)x}, x \neq \frac{1}{1-ab}.$$

From expression above we have that the critical curve LC_{-1} is given by

$$LC_{-1} = \left\{(x, y) \in \mathbb{R}^2_+ : y = \frac{1-x}{1-(1-ab)x}, x \neq \frac{1}{1-ab}\right\}, \tag{4.3}$$

and it is formed by two branches:

(i) LC^1_{-1}, a curve connecting the points $(0, 1)$ and $(1, 0)$, for $x < \frac{1}{1-ab}$.

(ii) LC^2_{-1}, an unbounded curve for $x > \frac{1}{1-ab}$.

These two connected components of LC_{-1} divide the domain $\mathbb{R}^2_+$ in the following regions.

$$\mathcal{R}_1 = \left\{(x,y) \in \mathbb{R}^2_+ : y \le \frac{1-x}{1-(1-ab)x} \text{ and } x < \frac{1}{1-ab}\right\},$$
$$\mathcal{R}_3 = \left\{(x,y) \in \mathbb{R}^2_+ : y \ge \frac{1-x}{1-(1-ab)x} \text{ and } x > \frac{1}{1-ab}\right\}, \text{ and}$$
$$\mathcal{R}_2 = (\mathbb{R}^2_+ \backslash \mathcal{R}_1 \cup \mathcal{R}_3)$$

Away from the critical curves, every point in $\mathbb{R}^2_+$ is a regular point and in each region $\mathcal{R}_i$, the determinant of the Jacobian does not change sign. In fact, we have the following result.

Lemma 4.1. *The following statements hold true for the determinant of the Jacobian of the map F:*

1. $J(x,y) > 0$ *for all* $(x,y) \in \mathcal{R}_1 \backslash LC^1_{-1}$*,*
2. $J(x,y) < 0$ *for all* $(x,y) \in \mathcal{R}_2 \backslash LC_{-1}$*,*
3. $J(x,y) > 0$ *for all* $(x,y) \in \mathcal{R}_3 \backslash LC^2_{-1}$*.*

Proof. Let us consider the three regions separately.

1. Let $(x,y) \in \mathcal{R}_1 \backslash LC^1_{-1}$. Then $0 \le x < 1$ and $y < \frac{1-x}{1-(1-ab)x}$. Since $0 < ab < 1$, it follows that $0 < 1-(1-ab)x < 1$. Hence from the relation $y < \frac{1-x}{1-(1-ab)x}$ we have that
$$y(1-(1-ab)x) - 1 + x < 0.$$
Consequently the sign in (4.2) is positive.

2. Let us define the sets
$$\mathcal{R}_{2,1} = \left\{(x,y) \in \mathbb{R}^2_+ : y > \frac{1-x}{1-(1-ab)x} \text{ and } x < \frac{1}{1-ab}\right\},$$
$$\mathcal{R}_{2,2} = \left\{(x,y) \in \mathbb{R}^2_+ : x = \frac{1}{1-ab} \text{ and } y \ge 0\right\}$$
and
$$\mathcal{R}_{2,3} = \left\{(x,y) \in \mathbb{R}^2_+ : y < \frac{1-x}{1-(1-ab)x} \text{ and } x > \frac{1}{1-ab}\right\}.$$
Then $\mathcal{R}_2 = \cup_{i=1}^3 \mathcal{R}_{2,i} \cup LC_{-1}$. We now subdivide this case in the following subcases.

(a) Let $(x, y) \in \mathcal{R}_{2,1}$. Since $(1-ab)x < 1$, then from the relation $y > \frac{1-x}{1-(1-ab)x}$ we have that $y(1-(1-ab)x) - 1 + x > 0$. Hence $J(x, y) < 0$.

(b) Let $(x, y) \in \mathcal{R}_{2,2}$. In this case it follows that

$$-1 + x + y(1-(1-ab)x) = \frac{ab}{1-ab} > 0.$$

Hence $J(x, y) < 0$.

(c) Analogously we show that $\det(J(x, y)) < 0$ if $(x, y) \in \mathcal{R}_{2,3}$.

3. The proof is similar to the above cases and will be omitted.

□

Remark 4.2. It follows from Lemma 4.1 that on $\mathcal{R}_1 \backslash LC_{-1}^1$ and $\mathcal{R}_3 \backslash LC_{-1}^2$ the map F is orientation preserving, while it is orientation reversing on $\mathcal{R}_2 \backslash LC_{-1}$.

We summarize our notation and results in Figure 3 where we depict the relative position of these three regions and the sign of the determinant of the Jacobian map.

Since our interest is to understanding the Ricker Map and its dynamics, we will be concerned with iterations of F. In particular, we will look at the dynamics of the images of the critical curves. Let us now give the the definition of critical curves of any rank.

Definition 4.3. The critical curves of rank k are the images of rank k of LC_{-1} denoted by

$$LC_{k-1} = F^k(LC_{-1}), k = 0, 1, 2, \ldots$$

For the Ricker map F given in (2.3), the curve LC_0 is formed by two branches:

(i) The curve $LC_0^1 = F(LC_{-1}^1)$ in which the end points are $(e^{r-1}, 0)$ and $(0, e^{s-1})$.

(ii) The curve $LC_0^2 = F(LC_{-1}^2)$.

A prototype of the relative positions of the curves LC_{-1}, LC_0, and LC_1 are depicted in Figure 4. We will show that for all $r, s > 0$, the prototypes in Figure 4 reveals the geometric information of the image of the Ricker map. More precisely, LC_0^2 is a curve lying below LC_0^1 and that the curve LC_0^1 bounds the image of F. Hence the map F reaches its maximum value on LC_0^1.

In order to formally establish the geometry of the image of the Ricker map, we will need tools from differential topology and some analytical analysis. Let us state the preliminary results we will need.

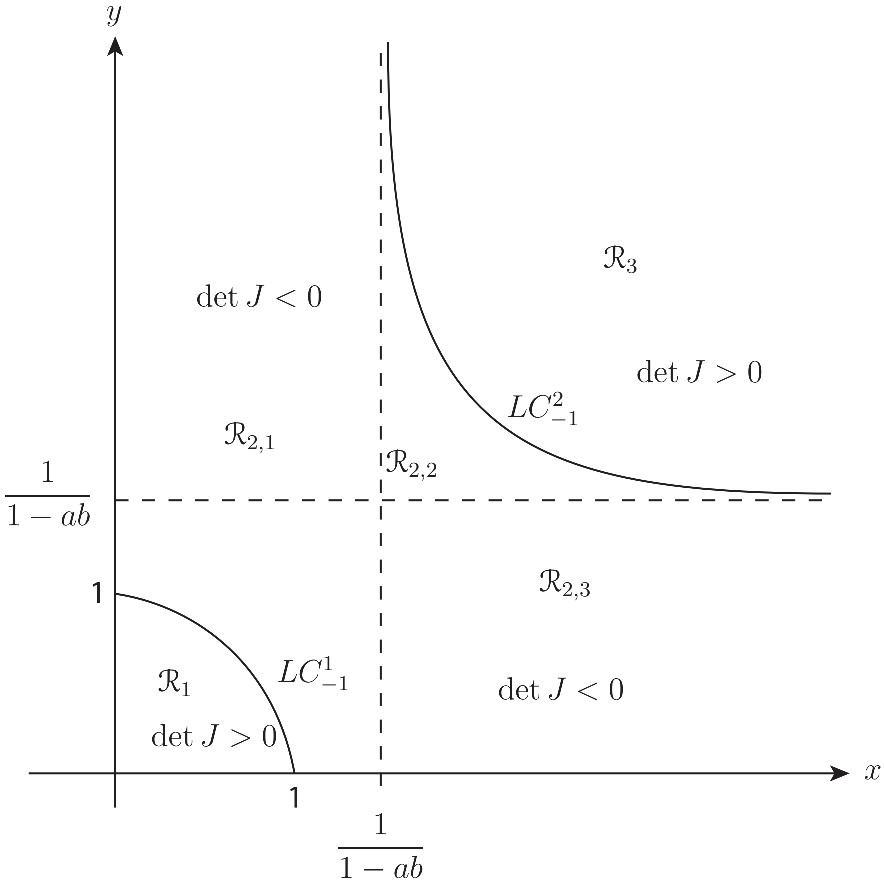

Figure 3: The subdivision of the positive quadrant into three connected regions $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ which are bounded by the critical curves LC^1_{-1} and LC^2_{-1}.

Lemma 4.4. *Let F be the Ricker map F given in* (2.3). *The following statements hold.*

(i) The x-axis and y-axis are invariant sets.

(ii) $\lim_{\|p\|\to\infty} F(p) = (0,0)$.

In particular, F has a continuous extension to the one-point compactification of $\mathbb{R}^2_+$, denoted by $\widehat{\mathbb{R}^2_+} = \mathbb{R}^2_+ \cup \{\infty\}$. Moreover, $F(\mathbb{R}^2_+)$ is compact.

Proof. Let us denote the x-axis and the y-axis by $X = \{(x,0)|\, x \geq 0\}$ and $Y = \{(0,y)|\, y \geq 0\}$, respectively. Clearly, if $p = (x,0) \in X$, then we have that $F(p) = (xe^{r-x}, 0) \in X$. Also, if $p = (x,y)$ and $F(p) \in X$, then it must be that $y = 0$.

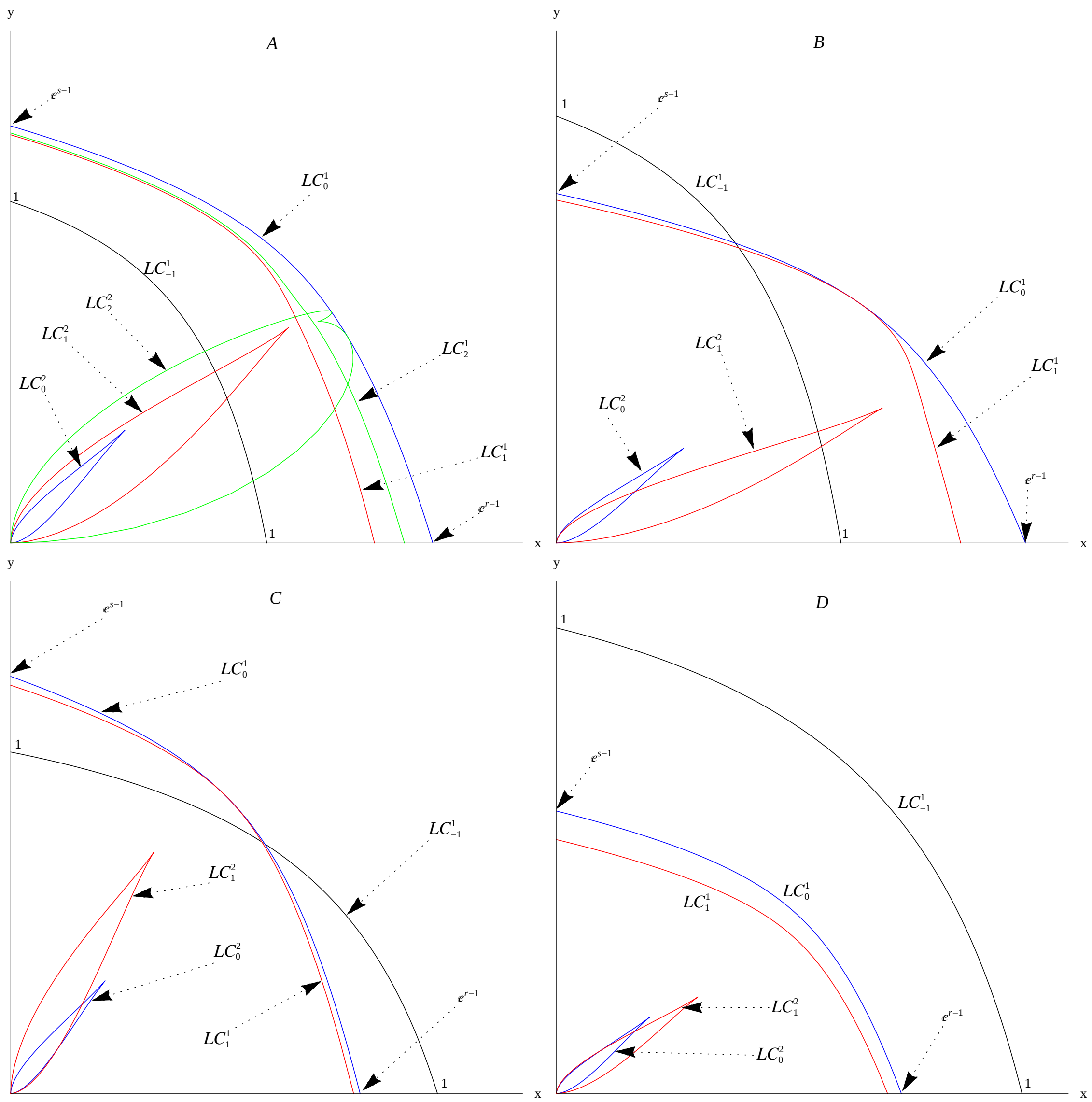

Figure 4: The relative position of the first critical curves LC_{-1}, LC_0, LC_1 and LC_2 according to the values of the carrying capacities r and s when $a = b = 0.5$. In plot A we have $r > 1$ and $s > 1$, in plot B one has $r > 1$ and $s < 1$, in plot C we have $r < 1$ and $s > 1$ and in plot D one has $r < 1$ and $s < 1$.

Therefore, X is an invariant set. By a similar argument, one may show that Y is also invariant.

Next, let $p = (x, y)$. If $\|p\| \to \infty$, then $x \to \infty$ or $y \to \infty$. Without loss of generality, let us assume that $x \to \infty$. Therefore,

$$\lim_{\|p\|\to\infty} F(p) = \lim_{x\to\infty} F(x,y) = \lim_{x\to\infty} (xe^{r-x-ay}, ye^{s-y-bx}) = (0,0) \tag{4.4}$$

as we desired.

Now, let $\widehat{\mathbb{R}^2_+} = \mathbb{R}^2_+ \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^2_+$, see [27] for the precise topological definitions. Let $\widehat{F} : \widehat{\mathbb{R}^2_+} \to \widehat{\mathbb{R}^2_+}$ be the map given by

$$\widehat{F}(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in \mathbb{R}^2_+ \\ (0,0), & \text{if } (x,y) = \infty. \end{cases}$$

From equation (4.4), $\widehat{F}$ is continuous and we conclude that $\widehat{F}\left(\widehat{\mathbb{R}^2_+}\right)$ is compact. Since $\widehat{F}(\infty) = (0,0)$ and $(0,0) \in F(\mathbb{R}^2_+)$ since it is a fixed point of F, it follows that $\widehat{F}\left(\widehat{\mathbb{R}^2_+}\right) = F\left(\mathbb{R}^2_+\right)$ and hence $F(\mathbb{R}^2_+)$ is compact.

□

Lemma 4.5. *For the Ricker F given in* (2.3)*, we have*

$$\partial F\left(\mathbb{R}^2_+\right) \subseteq F\left(\partial \mathbb{R}^2_+\right) \cup LC_0$$

Proof. Let $q \in \partial F\left(\mathbb{R}^2_+\right)$, then because $F(\mathbb{R}^2_+)$ is compact, hence closed, there is $p \in \mathbb{R}^2_+$ with $F(p) = q$. Then we have that p is either a regular or a singular value of F.

First, let us assume that p is singular, then $p \in LC_{-1}$ and $q \in LC_0$. Next, suppose p is regular. We have two cases. If $p \in int(R^2_+)$, then there is a $\delta > 0$ such that $B(p,\delta) \subseteq R^2_+$, where $B(p,\delta)$ is the ball with radius δ centered at p. Since p is regular, there is $\varepsilon > 0$ such that $B(F(p),\varepsilon) = B(q,\varepsilon) \subseteq F\left(R^2_+\right)$, a contradiction as $q \in \partial F\left(R^2_+\right)$. Else, $p \in \partial \mathbb{R}^2_+$ and hence $q \in F\left(\partial \mathbb{R}^2_+\right)$.

□

Remark 4.6. This is a general result and its proof is also valid for maps of class C^1 on general compact Euclidean domains. Heuristically, Lemma. 4.5 states that the images of regular values cannot be on the boundary and any new boundary points must be images of singular points.

Theorem 4.7. *The Ricker map F, as given in* (2.3)*, is excellent.*

Proof. We will show that all the points in LC_{-1} are fold points except for one cusp point $\widehat{P}$ in LC^2_{-1}.

We will begin considering the first component of LC_{-1}, namely LC^1_{-1}. Using the definition of a fold point as given in (3.1), we now consider a parametrization of LC^1_{-1} given by a curve φ_1 defined as $\varphi_1 : [0,1] \to \mathbb{R}^2$, where

$$\varphi_1(t) = \left(t\,,\,\frac{1-t}{1-t+abt}\right)$$

We know from [25] that it suffices to show that for the parametrization φ_1 given above, we have

$$\frac{d}{dt}(F \circ \phi_1)(0) \neq 0.$$

Let $F \circ \varphi_1(t) = (\alpha_1(t), \alpha_2(t)) = \alpha(t)$. Thus, α is a curve from $\alpha(0) = (0, s)$ to $\alpha(1) = (r, 0)$. We will show that $\alpha_1'(t)$ and $\alpha_2'(t)$ do not vanish for $t \in [0, 1]$.

Using the parametrization of LC_{-1}^1 above, a direct computation yields:

$$\alpha'(t) = (\alpha_1'(t), \alpha_2'(t)) = (\rho_1(t)h(t), \rho_2(t)h(t))\,,$$

where

$$\begin{aligned}
\rho_1(t) &= -\frac{1}{(1-t+abt)^2} \cdot e^{\frac{r-rt+rabt-t+t^2-abt^2+ta-a}{1-t+abt}}\,, \\
\rho_2(t) &= \frac{b}{(1-t+abt)^3} e^{\frac{s-st+sabt-1+t-bt+bt^2-b^2t^2a}{1-t+abt}}\,, \text{ and}
\end{aligned}$$

$$h(t) = (ab-1)^2\,t^3 + \left(-3 - a^2b^2 + 4\,ab\right)t^2 + \left(-2\,ab + 3 - a^2b\right)t - 1\,.$$

Therefore, since $\rho_1(t)$ and $\rho_2(t)$ do not vanish, in order for $\alpha_1'(t)$ or $\alpha_2'(t)$ to be equal zero, we must have $h(t) = 0$. We will now show that the cubic polynomial $h(t)$ does not have roots on the interval $[0, 1]$. Indeed, we can expand $h(t)$ and manipulate it as follows,

$$\begin{aligned}
h(t) &= -1 + (-2ab + 3 - a^2b)t + (-3 - a^2b^2 + 4ab)t^2 + (ab-1)^2t^3 \\
&= -a^2bt - (1-t)(1 + ((ab-1)t)^2)
\end{aligned}$$

Therefore, $h(t) = -a^2bt - (1-t)(1 + ((ab-1)t)^2) < 0$ for all $0 \leq t < 1$ and for $t = 1$, $h(1) = -a^2b < 0$. Hence $h(t)$ does not vanish in $[0, 1]$ and all points in LC_{-1}^1 are fold points as we claimed.

Now, we will do a similar analysis for LC_{-1}^2 and show that all points, except one, are fold points and the exception is a cusp point.

Consider a parametrization of LC_{-1}^2 given by a curve $\varphi_2 : (0, 1) \to \mathbb{R}^2$ with

$$\varphi_2(t) = \left(\frac{1}{(1-ab)\,t}, \frac{(1-ab)t-1}{(1-ab)(1-t)} \right)$$

Now let $F \circ \varphi_2(t) = (\beta_1(t), \beta_2(t)) = \beta(t)$. From (ii) in Lemma 4.4, we have

$$\lim_{t\to 0} \beta(t) = \lim_{t\to 1} \beta(t) = (0,0).$$

Also, for any $t \in (0,1)$, we have that $\beta(t) \in int(\mathbb{R}^2_+)$. Thus β is a curve in the first quadrant that begins and ends at the origin, hence it must change direction at least once, that is, there is t_0 such that $\beta'(t_0) = 0$.

A direct computation shows that

$$\beta'(t) = (\beta_1'(t), \beta_2'(t)) = (\rho_1(t)h(t), \rho_2(t)h(t)),$$

$$\rho_1(t) = -\frac{1}{(1-ab)^2\, t^3\, (t-1)^2} \cdot e^{\frac{-rt^2+rt+rt^2ab-rtab+t-1+at^2-a^2t^2b-at}{(-1+ab)t(t-1)}},$$

$$\rho_2(t) = \frac{b}{(1-ab)^2\, t^2\, (t-1)^3} \cdot be^{\frac{-st^2+st+st^2ab-stab+t^2-t^2ab-t+tb-b}{(-1+ab)t(t-1)}}, \text{ and}$$

$$h(t) = (1-ab)t^3 + (2ab + a^2b - 3)t^2 + (3-ab)t - 1\ .$$

It is clear that $\rho_1(t)$ and $\rho_2(t)$ do not vanish in $(0,1)$, thus if $\beta_1'(t)$ or $\beta_2'(t)$ is equal to zero, then both are.

We now claim that the cubic polynomial $h(t)$ has exactly one root of multiplicity one in the interval $(0,1)$. Indeed, since $\beta(t)$ is continuous, it follows by our observation above that the curve β must change directions, we must have that $h(t)$ has a root of odd degree in $t_0 \in (0,1)$. This can also be analytically verified as $h(0) = -1 < 0$ and $h(1) = a^2b > 0$.

Suppose towards a contradiction that either t_0 has multiplicity greater than one or there are other roots of $h(t)$ in $(0,1)$. A simple analysis then shows that h has an inflection point in $(0,1)$, that is, $h''(t) = 0$ has a solution in $(0,1)$. A computation yields the solution of $h''(t) = 0$ to be

$$t = \frac{3 - 2ab - a^2b}{3(1-ab)} \in (0,1).$$

From the inequalities above, we obtain two conditions on a, b

(i) $b < \dfrac{3}{a(2+a)}$ and

(ii) $a > 1$.

Next, since $h'(t)$ must have solutions in $(0,1)$, it must be that its discriminant is nonnegative. In other words,

$$\Delta = a^4b^2 - 6a^2b + 4a^3b^2 + a^2b^2 = a^2b(a^2b - 6 + 4ab + b) \geq 0$$

Using (i) above, we have that

$$\begin{aligned} a^2b - 6 + 4ab + b &< \frac{3a^2}{a(2+a)} - 6 + \frac{4a}{a(2+a)} + \frac{3}{a(2+a)} \\ &< \frac{-3(a-1)(a+1)}{a(a+2)} \\ &< 0 \end{aligned}$$

since by (ii) $a > 1$. This is a contradiction and therefore there is a unique point $t_0 \in (0,1)$ where $h(t_0) = 0$ with $h'(t_0) \neq 0$. Thus

$$\frac{d}{dt}(F \circ \varphi_2)(t) \neq 0,$$

for all $t \neq t_0$ and

$$\frac{d}{dt}(F \circ \varphi_2)(t_0) = 0 \text{ and } \frac{d^2}{dt^2}(F \circ \varphi)(t_0) \neq 0.$$

This shows that every point of LC^2_{-1} is a fold except for $\widehat{P} = \varphi_2(t_0)$ which is a cusp point. This shows that the Ricker map is an *excellent* map.

$\square$

One interesting geometric consequence of the computation in Theorem 4.7 is the following result.

Corollary 4.8. *The location of the curve LC^2_0 is completely determined by the cusp point $\widehat{P}$, in fact, we have that*

$$LC^2_0 \subseteq [0, f_1(\widehat{P})] \times [0, f_2(\widehat{P})]$$

Proof. From the computation done in Theorem 4.7, we have that

(i) For $0 < t < t_0$ we have that $\beta_1'(t)$ and $\beta_2'(t)$ are negative, and

(ii) For $t_0 < t < 1$ we have that $\beta_1'(t)$ and $\beta_2'(t)$ are positive.

Hence, $\beta_1(t_0) \leq \beta_1(t)$ and $\beta_2(t_0) \leq \beta_2(t)$ for all $t \in (0,1)$. From which we conclude that

$$LC_0^2 \subseteq [0, f_1(\widehat{P})] \times [0, f_2(\widehat{P})] \ .$$

$\square$

Theorem 4.9. *In the regions $\mathcal{R}_1$ and $\mathcal{R}_3$, the restriction of the Ricker map F, as given in* (2.3)*, is injective.*

Proof. We start with $F_1 = F|_{\mathcal{R}_1}$ and will will establish that it is injective by showing that F_1 satisfies the conditions of Theorem 3.7. In particular, we will show that F_1 is injective when restricted to the boundary. Let us denote the boundary of $\mathcal{R}_1$ as follows.

$$\partial\overline{\mathcal{R}_1} = X \cup Y \cup LC_{-1}^1$$

where $X = [0,1] \times \{0\}$ and $Y = \{0\} \times [0,1]$, that is, the x-axis and y-axis restricted to $\overline{\mathcal{R}_1}$. First, it is easy to see that $F_1|_X$ and $F_1|_Y$ are injective since $F_1|_X$ and $F_1|_Y$ are increasing functions over $[0,1]$. Next, from (i) in Lemma. 4.4, $F_1(X) \cap F_1(Y) = F_1(X \cap Y) = \{(0,0)\}$. Similarly,

$$F_1(X) \cap F_1(LC_{-1}^1) = F_1(X \cap LC_{-1}^1) \text{ and } F_1(Y) \cap F_1(LC_{-1}^1) = F_1(Y \cap LC_{-1}^1)$$

Hence, it suffices to show that $F_1|_{LC_{-1}^1}$ is injective. From the analytic work we developed in the proof of Theorem 4.7 we see that the image of LC_{-1}^1 is a graph over the x-axis and y-axis. Indeed, we established that $\alpha'(t) \neq 0$ and all the points in LC_{-1}^1 are fold points. From Theorem 3.4, we know the local structure at each fold point and we conclude that F_1 is locally injective on $\mathcal{R}_1$. The other points in $\mathcal{R}_1$ are regular points of F_1, hence the map is locally injective. Finally, we conclude that F_1 is injective.

Now, let us consider $F_3 = F|_{\mathcal{R}_3}$. In a similar manner we will show that F_3 is injective by showing that F_3 satisfies the conditions of Theorem 3.7. However, Theorem 3.7 is only valid for compact sets. Therefore, we must consider the one point compactification of $\mathbb{R}_+^2$, denoted by $\widehat{\mathbb{R}_+^2}$. By Lemma 4.4, we have that F has a continuous extension to $\widehat{F} : \widehat{\mathbb{R}_+^2} \to \widehat{\mathbb{R}_+^2}$. Therefore, without loss of generality, we may view that $\mathcal{R}_3$ as a compact set and after all we can apply Theorem 3.7 to establish injectivity. Observe that $\partial\mathcal{R}_3 = LC_{-1}^2 \cup \{\infty\}$ and we will show that $F_3|_{LC_{-1}^2}$ is injective.

Once again from the work done in Theorem 4.7, we have that every point of LC_{-1}^2 is a fold except for one cusp point at $\widehat{P}$. Therefore, from the structure Theorems 3.4

and 3.5 we see that F_3 is locally injective. Thus an application of Theorem 3.7 shows that F_3 is injective.

□

Remark 4.10. It is not hard to show that the map F is injective in the interior of $\mathcal{R}_1$ and $\mathcal{R}_3$. Indeed, one can use Theorem 5.1 in the next section (see Smith [24]). The novelty of our work is to show that injective persists on the boundary.

From our analysis so far, we can now finally show that the prototype of the image of the Ricker map is indeed as shown in Figure 4. More precisely, we have the following result.

Theorem 4.11. *Let $\mathcal{D}$ be the region enclosed by the curve LC_0^1 and the axes. Then $Im(F) = \mathcal{D}$.*

Proof. We know from Lemma 4.4 that the axes are invariant and $Im(F)$ is a compact set. Then $F\left(\partial\mathbb{R}_+^2\right) = X \cup Y$ where $X = \{(x,0)|\, 0 \le x \le e^{r-1}\}$ and $Y = \{(0,y)|\, 0 \le y \le e^{s-1}\}$.

Now, we claim that $F(\widehat{P}) \in intF\left(\mathbb{R}_+^2\right)$. Indeed, by the local structure at the cusp, given by Theorem 3.5, implies that $F(\widehat{P})$ is an exposed point in the image of F. In fact, there is an open ball around $\widehat{P}$ that maps to an open ball at $F(\widehat{P})$, as depicted in Figure 5, establishing that $F(\widehat{P}) \in intF\left(\mathbb{R}_+^2\right)$ and hence exposed in every direction.

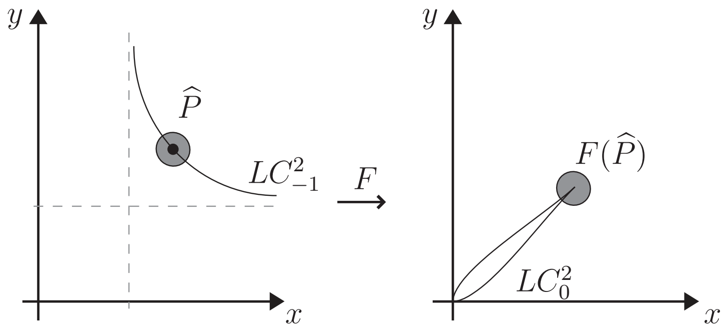

Figure 5: The cusp point in an interior point of the image, hence exposed.

Next, by Lemma 4.5 we have that

$$\partial F\left(\mathbb{R}_+^2\right) \subseteq X \cup Y \cup F(LC_{-1}) \,. \tag{4.5}$$

From the definition of an exposed point, any ray starting at $F(\widehat{P})$ must intersect $\partial F\left(\mathbb{R}_+^2\right)$. Using (4.5), in the direction of the first quadrant, a ray must intersect $\partial\mathcal{D}$, as depicted in Figure 4.

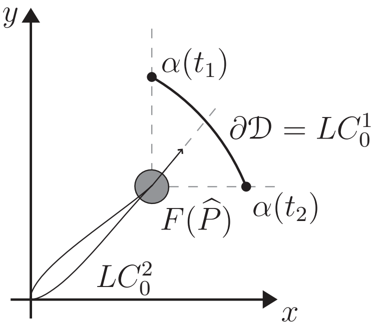

Figure 6: The rays emanating from the exposed cusp in the positive direction must intersect $\partial\mathcal{D}$.

Using the notation from Theorem 4.7, recall that the parametrization of LC_0^1 is a curve α. Then, let us denote the intersections of the ray starting at $F(\widehat{P})$ in the direction $v = (0,1)$ and $v = (1,0)$ by $\alpha(t_1)$ and $\alpha(t_2)$, respectively. We have shown, that $\alpha'(t) \neq 0$ which implies the following:

(i) For $0 \leq t \leq t_1$ we have that $\alpha_2(t) > f_2\left(\widehat{P}\right)$, and

(ii) For $t_2 \leq t \leq 1$ we have that $\alpha_1(t) > f_1\left(\widehat{P}\right)$.

Therefore LC_0^2 will remain below LC_0^1 and we will have

$$\partial F\left(\mathbb{R}_+^2\right) = X \cup Y \cup LC_0^1 .$$

□

5 Pre-image function

We start this section determining the cardinality of the pre-images of points in the codomain of the Ricker map. First, let us recall that a map is *proper* if the inverse image of a compact set is compact. The following result is well known in this field, see [3, p.27]

Theorem 5.1 (Chow-Hale, page 27 [3])**.** *Suppose X and Y are metric spaces, $F : X \to Y$ is continuous and proper, and, for each $y \in Y$, let $N(y)$ be the cardinal number of $F^{-1}(y)$. Then $N(y)$ is finite and constant on each connected component of $Y \backslash LC_0$.*

In our applications, we will consider the continuous extension $\widehat{F}$ of the Ricker map F which is clearly proper. Hence Theorem 5.1 applies to our model.

Observe that $\mathbb{R}^2_+ \backslash LC_0$ consists of three connected components $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, as depicted in Figure 8, and by Theorem 5.1, $F^{-1}(y)$ has constant finite cardinality on each component. Let us denote by Z_n the *zone* in $\mathbb{R}^2_+$ where the cardinality of $F^{-1}(y)$ is n.

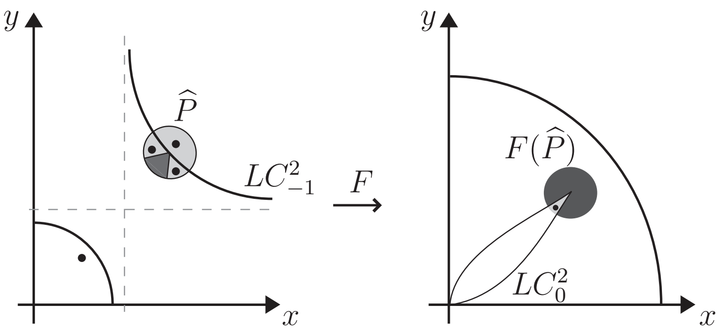

Figure 7: Number of pre-images of a point near the cusp point. The three points in the domain inside the gray neighborhood of the cusp in the left hand panel are mapped to the same point inside the gray region in the right hand panel.

From Theorem 4.11, we have that $\mathcal{C}_3$ is in zone Z_0. By Theorem 3.4 we know that a point in $\mathcal{C}_2$ has exactly two pre-images, i.e, $\mathcal{C}_2$ is in zone Z_2. Now, let $q \in \mathcal{C}_1$ be a point close to the cusp point (see Figure 5). Then, q has exactly four pre-images: one belongs to $\mathcal{D}$ with smaller x and smaller y than q and the other three are coming from the local structure near the cusp point by Theorem 3.5. Hence, we have that $\mathcal{C}_1$ is in zone Z_4.

From the dynamics of the one dimensional Ricker map, we see that points in the axes are in zone Z_2, except for the critical points. In fact, the critical curve LC^1_0 is contained in zone Z_1. Finally, by the local structure near the cusp point it follows that on LC^2_0 the map is contained in zone Z_3. In Figure 8 is presented a prototype of the number of pre-images of a point in each connected component.

The analysis of the Ricker map via its isoclines plays an important role and we now recall its definition. The isoclines of a map $F = (f, g)$ are defined as $f(x, y) = x$ and $g(x, y) = y$. In the Ricker competition map defined by (2.3) these are the lines $ay + x = r$ denoted by ℓ_1 and $y + bx = s$ denoted by ℓ_2. Moreover, the map F takes a point $(x, y) \in \mathbb{R}^2_+$ lying above (below) ℓ_1 to a point with a smaller (larger) $x-$coordinate. Similarly, the map F takes a point $(x, y) \in \mathbb{R}^2_+$ lying above (below) s_2 to a point with smaller (larger) $y-$coordinate. Note that on the isocline ℓ_1, the

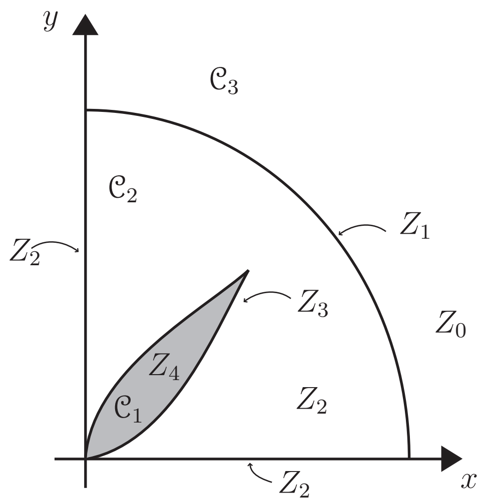

Figure 8: The set $\mathbb{R}^2_+ \backslash LC_0$ consists of three connected components $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. In each component, the number of pre-images of a point is constant and we denoted a zone by Z_i if this number is i. In addition, the arrows indicate the number of pre-images in the boundary.

population x has no growth, that is $x_{n+1} = x_n$ and on the isocline ℓ_2 the population y has no growth, that is $y_{n+1} = y_n$.

When the two isoclines ℓ_1 and ℓ_2 intersect in the positive quadrant, then the map has a coexistence fixed point. In the case of the Ricker competition model this point is given by

$$(x^*, y^*) = \left(\frac{r - as}{1 - ab}, \frac{s - br}{1 - ab}\right).$$

Remember that we are considering $ab < 1$ since when $ab > 1$ the asymptotic attractor of an orbit depends on its initial condition. The case $ab = 1$ is discarded since in this case the two isoclines are parallel and no coexistence fixed point is presented.

For convenience we divide the forward invariant region $\mathcal{D}$ into four regions Γ_1, Γ_2, Γ_3 and Γ_4 (see Figure 9) be as follows

$$\Gamma_1 = \{(x, y) \in \mathbb{R}^2_+ : y < -bx + s \text{ and } y < \frac{-x + r}{a}\},$$

$$\Gamma_2 = \{(x, y) \in \mathbb{R}^2_+ : \frac{-x + a}{a} \leq y \leq -bx + s \text{ and } y \leq LC^1_0\},$$

$$\Gamma_3 = \{(x, y) \in \mathbb{R}^2_+ : -bx + s \leq y \leq \frac{-x + a}{a} \text{ and } y \leq LC^1_0\},$$

and

$$\Gamma_4 = \{(x, y) \in \mathbb{R}^2_+ : y > -bx + s \text{ and } y > \frac{-x+a}{a} \text{ and } y \leq LC^1_0\}.$$

The direction of the orbits in each one of these four sets is shown by the directions

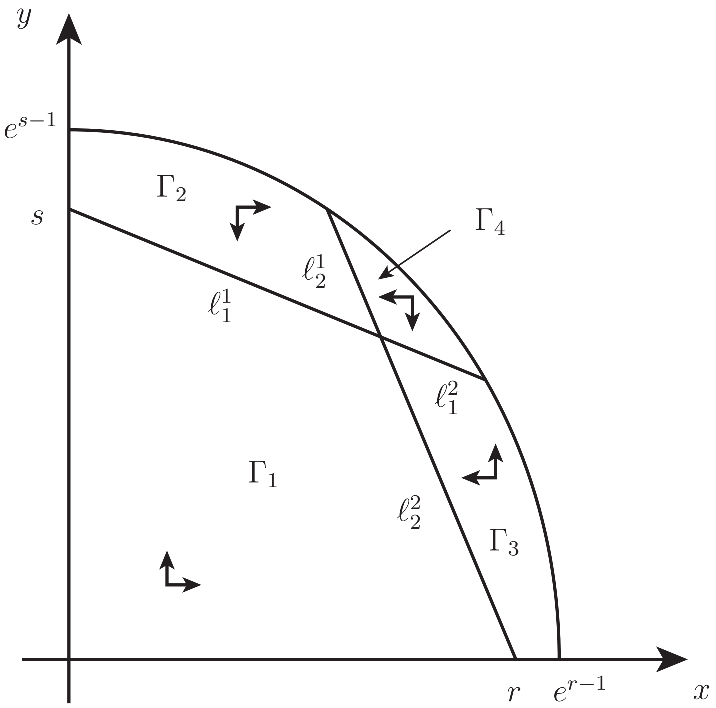

Figure 9: The subdivision of the invariant region $\mathcal{D}$ into four regions (Γ_1, Γ_2, Γ_3 and Γ_4). The intersection of the two isoclines indicates the existence of a positive fixed point. The arrows indicates the direction of the orbits in each one of the sets Γ_i, $i = 1, 2, 3, 4$.

of the arrows. For more details about this point see [17]. Let ℓ_1^1, ℓ_1^2, ℓ_2^1 and ℓ_2^2 be the line segments on the isoclines as it is shown in Figure 9. Notice that we require that

$$(x^*, y^*) \in \ell_i^j, i, j = 1, 2$$

and by geometrical construction we have that

$$\ell_1 = \ell_1^1 \cup \ell_1^2 \qquad \text{and} \qquad \ell_2 = \ell_2^1 \cup \ell_2^2.$$

We now state the following lemma

Lemma 5.2. *The following are true.*

1. *if* $(x,y) \in \ell_1^1$ *then either* $F(x,y) = (0,s)$ *or* $F(x,y) \in \Gamma_2 \cup \Gamma_4 \backslash \ell_1$*,*

2. *if* $(x,y) \in \ell_2^1$ *then* $F(x,y) \in \Gamma_1 \cup \Gamma_2 \backslash \ell_2$*,*

3. *if* $(x,y) \in \ell_2^2$ *then either* $F(x,y) = (r,0)$ *or* $F(x,y) \in \Gamma_3 \cup \Gamma_4 \backslash \ell_2$*,*

4. *if* $(x,y) \in \ell_1^2$ *then* $F(x,y) \in \Gamma_1 \cup \Gamma_3 \backslash \ell_1$*.*

Proof. We present the proof for the first case and the other cases can be investigated in a similar way. On the line segment isocline ℓ_1^1, if $x = 0$ then $F(x,y) = (0,s)$, if $(x,y) = (x^*,y^*)$ then $F(x,y) = (x^*,y^*)$. If not, since y is fixed and x is moving to the right, it follows that $F(x,y) \in \Gamma_2 \cup \Gamma_4 \backslash \ell_1$. □

Since the image of $\mathcal{R}_1$ is the region $\mathcal{D}$ we will define the principal pre-image function as

$$F_p^{-1} : \mathcal{D} \to \mathcal{R}_1. \tag{5.1}$$

Due the fact that $\mathcal{C}_2$ is zone Z_2, a secondary pre-image may exists on $\mathcal{D}\backslash\mathcal{R}_1$. Hence, we define the secondary pre-image function as

$$F_s^{-1} : \mathcal{D} \to \mathcal{D}\backslash\mathcal{R}_1. \tag{5.2}$$

Notice that both F_p^{-1} and F_s^{-1} are continuous.

Using the pre-image function we prove the lemma below.

Lemma 5.3. *The following statements are true.*

1. *If* $(x,y) \in \Gamma_3$ *then* $F(x,y) \notin \Gamma_2$ *with the exception of the fixed point* (x^*,y^*)*.*

2. *If* $(x,y) \in \Gamma_2$ *then* $F(x,y) \notin \Gamma_3$ *with the exception of the fixed point* (x^*,y^*)*.*

Proof. Since the boundary of Γ_3, $\partial\Gamma_3$, is a compact set and F is continuous, we have that $F(\partial\Gamma_3)$ is a compact set. Now

$$F(0, e^{s-1}) = \exp(2s - 1 - e^{s-1}) \begin{cases} > s \text{ if } s < 1 \\ = 1 \text{ if } s = 1 \\ < s \text{ if } s > 1 \end{cases}.$$

Therefore, the image of the line segment $\Upsilon = \{y : s \le y \le e^{s-1}\}$ on the $y-axis$, is a segment Υ' on the $y-axis$ connecting the points $(0,s)$ and $(0, \exp(2s-1-e^{s-1}))$ either above or below $(0,s)$. By Lemma 5.2, $\ell_1^{1'} = F(\ell_1^1)$ is a curve connecting the points $(0,s)$ and (x^*,y^*) passing on the set $\Gamma_3 \cup \Gamma_4$ and $\ell_2^{1'} = F(\ell_2^1)$ is a curve connecting the points (x^*, y^*) and $(x_p, y_p) \in \Gamma_1 \cup \Gamma_3$, lying on the set $\Gamma_1 \cup \Gamma_3$. Consequently, the image of the curve $lc_0 = LC_0^1 \cap \Gamma_3$ is the curve lc_0' connecting the points $(0, \exp(2s-1-e^{s-1}))$ and (x_p, y_p) (since $F(\partial\Gamma_3)$ is a compact set). Hence, $F(\partial\Gamma_3)$ is the closed curve $\Upsilon' \ell_1^{1'} \ell_1^{1'} lc_0'$. Moreover, if $X \in F(\partial\Gamma_3)$, then $X \notin \Gamma_2$ with the exception of the positive fixed point. Notice that, the image of the boundary of Γ_3 is the boundary of $F(\Gamma_3)$.

Finally, we will show that $F(\Gamma_3) \cap \Gamma_2 = \emptyset$. If not there exists $(x_0, y_0) \in int\,(\Gamma_3)$ such that $F(x_0, y_0) = (\widehat{x}_0, \widehat{y}_0) \in \Gamma_2$ and $F(x_0, y_0) \neq (x^*, y^*)$. Let $B = F(\Gamma_3) \cap \Gamma_2 \neq \emptyset$. Then B is compact and $\partial B \neq \emptyset$. Let $q \neq (x^*, y^*) \in \partial B$, then $F_s^{-1}(q) \in int\,(\Gamma_3)$. Since the secondary pre-image function is continuous, for some $\delta > 0$, $F_s^{-1}(\mathcal{B}(q, \delta)) \subset int(\Gamma_3)$, a contradiction.

The second part of the Lemma can be similarly established. □

6 Main result

In this section, we are focused on the global dynamics of the positive (coexistence) equilibrium point of the Ricker competition model (2.2). In particular, we prove that if $ab < 1$, and the coexistence equilibrium is locally asymptotic stable, then it is globally asymptotic stable provided that certain conditions are satisfied. Below we state our main result.

Theorem 6.1. *Let F be the Ricker competition model (2.2) with $1 < r, s < 2$. Suppose that the coexistence fixed point $X^* = \left(\frac{aL-K}{ab-1}, \frac{bK-L}{ab-1}\right)$ of F is locally asymptotically stable. Assume the following conditions:*

1. *The region $\mathcal{R}_1$ is a contained in the region Γ_1.*
2. *For all $m \neq n$, $LC_m^1 \cap LC_n^1 = \emptyset$.*

Then X^* is globally asymptotically stable with respect to the interior of the first quadrant.

The proof of Theorem 6.1 now proceeds in a series of Theorems and Lemmas. We will utilize a mixture of tools and ideas from geometry, topology, and analysis. Before we embark in the proof, we shall make a few remarks about our assumptions.

The hypotheses that the carrying capacities satisfy $1 \leq r, s \leq 2$, ensures that the dynamics on the positive axes is known. Indeed, for the one-dimensional Ricker equation $x_{n+1} = x_n e^{r-x_n}, n \in \mathbb{Z}^+$, we know that r is a globally asymptotically stable fixed point whenever $r \leq 2$.

The assumption that $\mathcal{R}_1$ is a contained in the region Γ_1, can be verified through the competition parameters as we will show in Lemma 6.2. Lastly, the assumption that images of critical curves do not intersect is motivated from the evidence in simulations, as depicted in Figure 10.

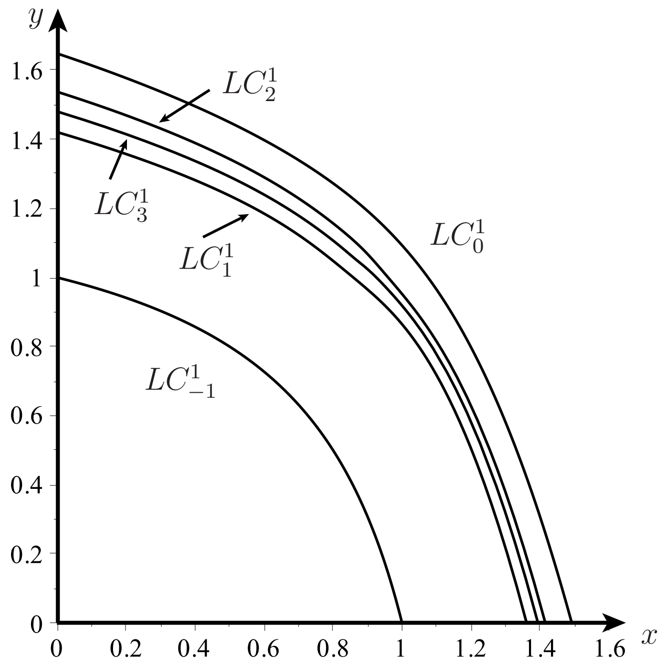

Figure 10: The relative position of the first critical curves LC_{-1}^1, LC_0^1, LC_1^1, LC_2^1, and LC_3^1 in a generic simulation. The values of the parameters in this example are $a = b = 0.5$, $r = 1.4$, and $s = 1.5$.

Let us now show how one can verify that $\mathcal{R}_1 \subset \Gamma_1$. Recall that the boundary of $\mathcal{R}_1$ is given by the critical curve LC_{-1}^1. Hence, $\mathcal{R}_1$ is a subset of Γ_1 whenever the systems

$$\begin{cases} y = \frac{1-x}{1-(1-ab)x} \\ ay = r - x \end{cases} \quad \text{and} \quad \begin{cases} y = \frac{1-x}{1-(1-ab)x} \\ y = s - bx \end{cases}$$

do not have real solutions. This happens in the first system when

$$-4(-1+ab)(a-r) + (1-a+r-abr)^2 < 0 \tag{6.1}$$

and in the second system when

$$-4b(-1+ab)(1-s) + (-1+b+s-abs)^2 < 0. \tag{6.2}$$

A straightforward calculation shows that Inequality (6.1) is satisfied whenever we have $\frac{a+1-2a\sqrt{b}}{1-ab} < r < \frac{a+1+2a\sqrt{b}}{1-ab}$. Similarly, Inequality (6.2) holds true whenever we have $\frac{1+b-2b\sqrt{a}}{1-ab} < s < \frac{1+b+2b\sqrt{a}}{1-ab}$. In fact, this can be summarized in the following result.

Lemma 6.2. *If the carrying capacities r and s satisfy $r \in \left(\frac{a+1-2a\sqrt{b}}{1-ab}, \frac{a+1+2a\sqrt{b}}{1-ab}\right)$ and $s \in \left(\frac{1+b-2b\sqrt{a}}{1-ab}, \frac{1+b+2b\sqrt{a}}{1-ab}\right)$, then the region $\mathcal{R}_1$ is contained in Γ_1.*

We remark that, there are values for the parameters a and b, where $\mathcal{R}_1$ is not contained in Γ_1. For instance, this is the case in Figure 11.

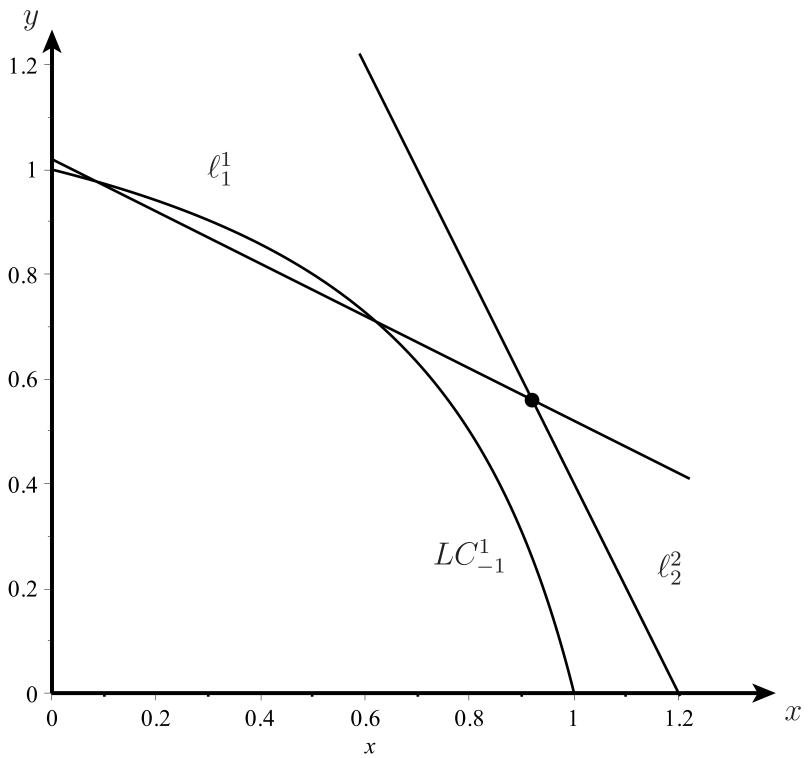

Figure 11: An example showing that $\mathcal{R}_1$ (region below LC_{-1}^1) is not contained in Γ_1 (region below the isoclines ℓ_1^1 and ℓ_2^2). In this case, the values of the parameters are $a = 0.5$, $b = 0.5$, $r = 1.2$, and $s = 1.02$.

As aforementioned, the proof of our main result will utilize a mixture of several ideas. One of the important concepts we will need is the notion of unstable manifolds. The locally unstable manifolds guarantee (see for instance [6, 14, 26]) that there exists a unique local unstable manifold $W_l^u(r) \subset \Gamma_3$ which is tangent to the eigenvector orthogonal to $\mathbf{x} = (r, 0)$. Similarly, there exists a unique local unstable manifold

$W_l^u(s) \subset \Gamma_2$ which is tangent to the eigenvector orthogonal to $\mathbf{y} = (0, s)$. In the sequel, we compute these two invariant sets. For details about such computations, see for instance [17].

After shifting the exclusion fixed point $(r, 0)$ to the origin, the Ricker competition model is now equivalent to

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \widetilde{f}(x_n, y_n) \\ \widetilde{g}(x_n, y_n) \end{bmatrix}$$

where

$$\begin{aligned} \widetilde{f}(x, y) &= (x + r)e^{r-(x+r)-ay} - r - (1 - r)x + ary \\ \widetilde{g}(x, y) &= ye^{s-y-b(x+r)} - e^{s-br}y, \end{aligned}$$

and the Jacobian at $(0, 0)$ is given by

$$J(0, 0) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} 1 - r & -ar \\ 0 & e^{s-br} \end{bmatrix}.$$

The diagonal matrix can be given by

$$\begin{bmatrix} 1 - r & -ar \\ 0 & e^{s-br} \end{bmatrix} = \begin{bmatrix} 1 & S_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 - r & 0 \\ 0 & e^{s-br} \end{bmatrix} \begin{bmatrix} 1 & \widetilde{S}_{12} \\ 0 & 1 \end{bmatrix},$$

where

$$S_{12} = -\widetilde{S}_{12} = \frac{-ar}{r - 1 + e^{s-br}}.$$

Using a new change of variables $x = u + S_{12}v$ and $y = v$, yields the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 1 - r & 0 \\ 0 & e^{s-br} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \widetilde{\widetilde{f}}(u_n, v_n) \\ \widetilde{\widetilde{g}}(u_n, v_n) \end{bmatrix},$$

where

$$\begin{bmatrix} \widetilde{\widetilde{f}}(u, v) \\ \widetilde{\widetilde{g}}(u, v) \end{bmatrix} = \begin{bmatrix} 1 & \widetilde{S}_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{f}(u + S_{12}v, v) \\ \widetilde{g}(u + S_{12}v, v) \end{bmatrix}.$$

Let $u = \Phi_1(v)$, where $\Phi_1(v) = p_0v + p_1v^2 + p_2v^3 + p_3v^4$. The function $\Phi_1(v)$ must satisfy the following equation

$$\Phi_1(e^{s-br}v + \widetilde{\widetilde{f}}(\Phi_1(v), v)) - (1 - r)\Phi_1(v) - \widetilde{\widetilde{f}}(\Phi_1(v), v) = 0. \tag{6.3}$$

After simplify this equation, we write the Taylor expansion and then we find the values of p_i, $i = 0, 1, 2, 3$. (See Appendix 7).

Rewriting in the original variables and back again to the original fixed point, the local unstable manifold $W_l^u(r)$ of the exclusion fixed point $(r, 0)$ is given by

$$W_l^u(r) = \{(x, y) \in \mathcal{D} : x = -S_{12}y + \Phi_1(y) + r\}.$$

Following the same ideas on can show that the local unstable manifold $W_l^u(s)$ of the exclusion fixed point $(0, s)$ is given by

$$W_l^u(s) = \{(x, y) \in \mathcal{D} : y = \frac{x}{Q_{11}} + \Phi_2\left(\frac{x}{Q_{11}}\right) + s\}, \tag{6.4}$$

where $Q_{11} = \frac{1-s-e^{r-as}}{bs}$ and $\Phi_2(x) = q_0x^2 + q_1x^3$. The values of q_0 and q_1 are in Appendix 7.

Since the coexistence equilibrium X^* is locally asymptotically stable, then there are two locally asymptotically stable manifolds tangent to the coexistence equilibrium: the local "**slow**" asymptotically stable manifold associated to the big eigenvalue (in absolute value) and the local "**fast**" asymptotically stable manifold associated to the small eigenvalue (in absolute value). Since the slow manifold plays a central rule here, we will focus our analysis in the local slow asymptotically stable manifold that we represent by $W_l^s(X^*)$.

The computations of the set $W_l^s(X^*)$ are long and we are not able to write it explicitly for general parameters as we did for the sets $W_l^u(r)$ and $W_l^u(s)$. Nevertheless, we are able to do it numerically. For instance, when $r = 1.5$, $s = 1.2$ and $a = b = 0.5$, the set $W_l^s(X^*)$ is given by

$$W_l^s(X^*) = \{(x, y) \in \mathcal{D} : G(x, y) = 0\}. \tag{6.5}$$

$G(x, y)$ is given in the Appendix.

Let a_x be an arc on $W_l^s(X^*)$ lying in Γ_2 such that $X^* \notin a_x$. Then $F_s^{-1}(s_r)$ is an arc. Since the computations here are long, using computer assistance one can see that $F_s^{-N}(a_x) \subset W_l^u(r)$, for some $N > 0$. Moreover, $F_s^{-n}(a_x) \to (r, 0)$ as $n \to \infty$. Furthermore, $F_s^{-n}(a_x \cup X^*)$, as $n \to \infty$, is an arc on $\mathcal{D}$ connecting the coexistence fixed point and the exclusion fixed point $(r, 0)$. This arc, in fact, is an approximation of the global unstable manifold of the exclusion fixed point $(r, 0)$ which we represent by $W_g^u(r)$). Hence, we assume that

$$F_s^{-n}(a_x \cup X^*) \subseteq W_g^u(r) \text{ as } n \to \infty.$$

Similarly, we will have

$$F_s^{-n}(a_y \cup X^*) \subseteq W_g^u(s) \text{ as } n \to \infty,$$

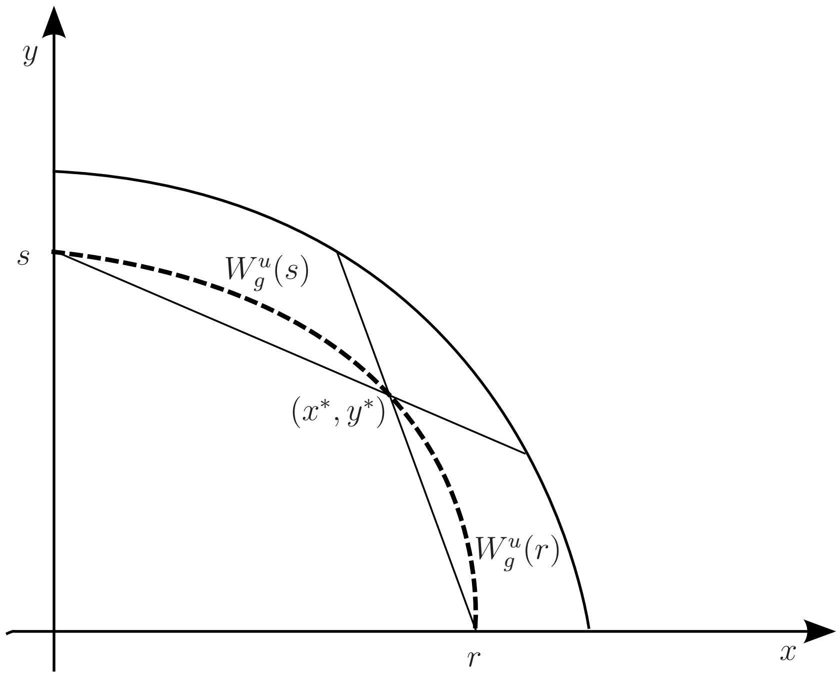

Figure 12: The existence of a globally unstable manifold of the exclusion fixed points $(r, 0)$ and $(0, s)$.

where a_y is an arc on $W_l^s(X^*)$ lying in Γ_3 and $W_g^u(s)$ is the global unstable manifold connecting the coexistence fixed point X^* and the exclusion fixed point $(0, s)$.

Notice that the sets $W_g^u(r)$ and $W_g^u(s)$ are invariant and $W_g^u(r) \cap W_g^u(s) = X^*$. For simplicity, let us write

$$W_g^u(r; s) = W_g^u(r) \cup W_g^u(s). \tag{6.6}$$

A point on the global unstable manifold $W_g^u(r; s)$ has two pre-images one of which belongs to $W_g^u(r; s)$ as this set is an invariant set. From the fact that the pre-image of an arc is an arc, it follows that $W_g^u(r; s)$ has two arcs as pre-images: $W_g^u(r; s) = F_s^{-1}(W_g^u(r; s))$ and $W_{-1}^{r;s} = F_p^{-1}\left(W_g^u(r; s)\right)$ lying below than $W_g^u(r; s)$.

Let $\mathcal{B}_1(X^*)$ be the immediate basin of attraction of X^* and define $\mathcal{B}_2(X^*) = F_s^{-1}(\mathcal{B}_1(X^*))$, $\mathcal{B}_3(X^*) = F_s^{-1}(\mathcal{B}_2(X^*))$, and interactively defining it as

$$\mathcal{B}_n(X^*) = F_s^{-1}(\mathcal{B}_{n-1}(X^*)).$$

Now, the union of all these sets is an open set lying around $W_g^u(r; s)$ which is part of the basin of attraction of the coexistence fixed point. Namely, define

$$\widehat{\mathcal{B}}(X^*) = \bigcup_{n=1}^{\infty} \mathcal{B}_n(X^*)$$

In other words, if $y \in \widehat{\mathcal{B}}(X^*)$, then $F^n(y) \to X^*$ as n goes to ∞. Hence we have the following result.

Proposition 6.3. *There exits an open set $\widehat{\mathcal{B}}(X^*)$ containing the globally unstable manifold which is in the basin of attraction of the coexistence fixed point of the Ricker competition model.*

Similar to the situation in one dimension, the convergence of the image of each point will be an alternating convergence where we will show that eventually it will be sufficiently close to the globally unstable manifold.

We are now ready to prove our main result.

Proof of Theorem 6.1. First, we observe that for any $p \in \mathcal{R}_1$, assumption (1) implies that $p \in \Gamma_1$. Since F is monotone in the region Γ_1, there must be some n such that $F^n(p) \in \mathcal{D}\backslash\mathcal{R}_1$. We will now show that any point in $\mathcal{D}\backslash\mathcal{R}_1$ not in the axes, will be globally attracted to X^*.

Let us introduce some notation that will help us in the presentation. From the condition that $1 \leq r, s \leq 2$, we know exactly how the dynamics of the one dimensional Ricker map works. In fact, let $r_{-1} = 1$ and for $m \geq 0$, denote $r_m = \pi_x\left(F^m(k_{-1}, 0)\right)$. Thus we know that $r_m \to r$ and $\{r_m\}$ is an alternating sequence converging to r, that is, the even and odd sequences are monotone. Similarly, we denote $\{s_m\}$ converging to s.

Let us define the region $\Omega_0 = \mathcal{D}\backslash\mathcal{R}_1$, that is the region bounded by the critical curves LC^1_{-1} and LC^{-1}_0 and the segments $\overline{r_{-1}r_0}$ and $\overline{s_{-1}s_0}$. In fact, let us define the region Ω_m, depicted in Figure 12, as

$$\partial\Omega_m = LC^1_{m-1} \cup LC^1_m \cup \overline{r_{m-1}r_m} \cup \overline{s_{m-1}s_m}$$

For any point $p \in \Omega_0$, we have that $F^m(p) \in \Omega_m$. Therefore it will suffice to show that for m sufficiently large, Ω_m will be contained in a ε-band around the unstable manifold $W^u_g(r; s)$. Then by Proposition 6.3 we will be done.

Indeed, from condition (2) we have that the sets $\{\Omega_m\}$ for a *properly* nested sequence of sets, that is,

$$\Omega_{m+1} \subsetneq \Omega_m$$

where we denoted properly nested to mean that a set is properly contained in the other. In fact, Ω_{m+1} is a proper subset of Ω_m. Observe that the image of Ω_m under F,

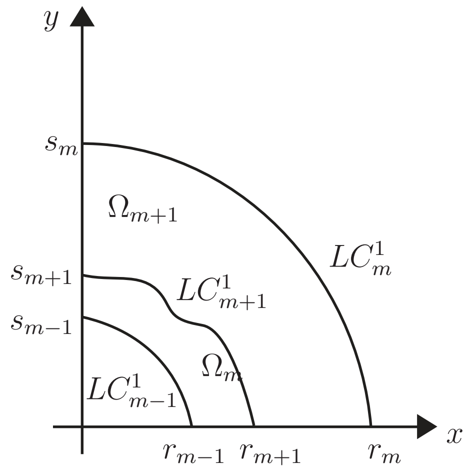

Figure 13: Regions bounded by the curves $F^n(LC^1_{-1})$

which is Ω_{m+1}, satisfies that on each of the axis, we know from the one-dimensional analyses that $\overline{r_m r_{m+1}} \subsetneq \overline{r_{m-1} r_m}$ and $\overline{s_m s_{m+1}} \subsetneq \overline{s_{m-1} s_m}$. Therefore, LC^{-1}_{m+1} must be a smooth path entirely contained in Ω_m and from (2) it cannot intersect the boundary of Ω_m. Thus,

$$\Omega_{m+1} \subsetneq \Omega_m$$

as we had claimed.

Now let us define

$$\Omega = \bigcap_{m=0}^{\infty} \Omega_m$$

We have that Ω is a nonempty and closed set. In fact, we will show that $\Omega = W^u_g(r; s)$. From its definition, we can say that the boundary of Ω is formed by two curves (possibly the same) γ^- and γ^+ connecting the points r to s on the axes. In addition, we see that the odd curves LC^1_{2i+1} are converging to γ^- and the even curves to γ^+.

Indeed, we have that for any $\varepsilon > 0$, there is m_0 such that for $2i \geq m_0$,

$$\begin{aligned} LC^1_{2i+1} &\subseteq N_\varepsilon(\gamma^-) \\ LC^1_{2i} &\subseteq N_\varepsilon(\gamma^+) \end{aligned}$$

Suppose towards a contradiction that $W^u_g(r; s) \neq \gamma^-$. Thus, we have that

$$\delta = \sup_{p\in\gamma} \left\{ \inf_{q\in\gamma^-} \{d(p,q)\} \right\} > 0$$

where $d(p,q)$ is the distance between the two points p and q. Now, for $\varepsilon = \dfrac{\delta}{2} > 0$, consider the ε-bands $N_\varepsilon(\gamma)$ and $N_\varepsilon(\gamma^-)$. Since $r, s \in N_\varepsilon(\gamma) \cap N_\varepsilon(\gamma^-)$, there is some $m \in \mathbb{N}$ such that $q_m^r \in B_\varepsilon(r)$ and $q_m^s \in B_\varepsilon(s)$. Thus we consider the paths α^r and α^s contained in LC_m^1 joining r to q_m^r and s to q_m^s, respectively. For simplicity of notation, let us consider just one of the paths above as denoted it by α.

Now let $\alpha_n = F^n(\alpha)$. By the local stability near $W_g^u(r;s)$, we must have $\alpha_n \subset N_\varepsilon(W_g^u(r;s))$ with its end point converging to X^*. On the other hand, $\alpha_n \subset LC_{m+n}^1 \subset N_\varepsilon(\gamma^-)$. However, by the choice of $\varepsilon = \frac{\delta}{2}$ the set $N_\varepsilon(W_g^u(r;s))$ is not contained in $N_\varepsilon(\gamma^-)$ and the ε-band near $W_g^u(r;s)$ will have to leave the ε-band near γ^- contradicting $\alpha_n \subset N_\varepsilon(\gamma^-)$. This is depicted in Figure 13.

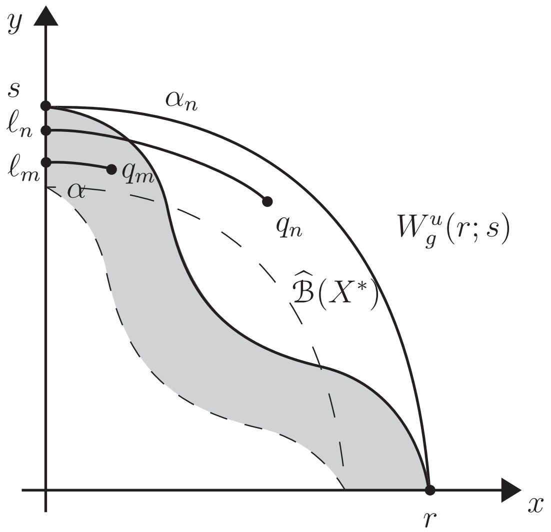

Figure 14: For n sufficiently large, path will approximate the unstable manifold, but must remain in shaded area, a contradiction.

A similar argument holds if $W_g^u(r;s) \neq \gamma^+$ by considering the even sequence of critical curves.

Therefore, we conclude that $\Omega = W_g^u(r;s)$. Thus choose $\varepsilon > 0$ as described in Proposition 6.3. Then for all $p \in int(\Omega_0)$, we have that for m sufficiently large, $F^m(p) \in N_\varepsilon(W_g^u(r;s))$, thus for n sufficiently large, $F^{m+n}(p) \to X^*$ establishing that

X^* is globally asymptotically stable with respect to the interior of the first quadrant. $\square$

We finalize the paper by stating an immediate consequence of Theorem 6.1. Using an analytic condition instead of a topological condition one can directly verify the first assumption in case of concrete values of the competition parameters a and b.

Corollary 6.4. *Let F be the Ricker competition model* (2.2) *with* $1 < r, s < 2$. *Suppose that the coexistence fixed point $X^* = \left(\frac{aL-K}{ab-1}, \frac{bK-L}{ab-1}\right)$ of F is locally asymptotically stable. Assume the following conditions:*

1. *The carrying capacities satisfy* $r \in \left(\frac{a+1-2a\sqrt{b}}{1-ab}, \frac{a+1+2a\sqrt{b}}{1-ab}\right)$ *and* $s \in \left(\frac{1+b-2b\sqrt{a}}{1-ab}, \frac{1+b+2b\sqrt{a}}{1-ab}\right)$.
2. *For all $m \neq n$, $LC_m^1 \cap LC_n^1 = \emptyset$.*

Then X^* is globally asymptotically stable with respect to the interior of the first quadrant.

Proof. Using Lemma 6.2, the proof follows directly from Theorem 6.1. $\square$

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7 Appendix

Values of p_i in the center manifold equation (6.3).

$$p_0 = 0,$$

$$p_1 = -\frac{ae^{2br}r\left(2e^s\left(e^s + e^{br}(-1+r)\right) + a\left(e^{2br} - e^{2s} - 2be^{br+s}r\right)\right)}{2\left(e^s + e^{br}(-1+r)\right)^2\left(e^{2s} + e^{2br}(-1+r)\right)},$$

$$p_2 = -\left(ae^{3br}r \left(\begin{array}{c} 3e^s\left(e^s+e^{br}(-1+r)\right)^2\left(3e^{2s}-e^{2br}(-1+r)\right)- \\ 6ae^s\left(e^s+e^{br}(-1+r)\right) \\ \left(\begin{array}{c} e^{3s}+(-1+b)e^{2br+s}r+3be^{br+2s}r+ \\ e^{3br}(-1-b(-1+r)r) \end{array} \right)+ \\ a^2\left(\begin{array}{c} e^{5s}+6be^{br+4s}r+e^{5br}(1+2r)- \\ 3be^{4br+s}r(3+b(-1+r)r)+ \\ e^{3br+2s}\left(-1-6br^2+6b^2r^2\right)+ \\ e^{2br+3s}\left(-1+(-2+3b)r+9b^2r^2\right) \end{array} \right) \end{array} \right)\right) /$$
$$\left(6\left(e^s+e^{br}(-1+r)\right)^3\left(e^{2s}+e^{2br}(-1+r)\right)\left(e^{3s}+e^{3br}(-1+r)\right)\right).$$

We omit the coefficient p_3 due the size of the expression.

Values of q_0 and q_1 for the center manifold function given by $\Phi_2(x)$ in (6.4).

$$q_0 = \frac{(-2+b)e^{2r}-be^{2as}+2e^{r+as}(1+(-1+ab)s)}{2b\left(e^{2r}+e^{2as}(-1+s)\right)s}$$

and

$$q_1 = \left(\begin{array}{c} (-3+b)^2e^{5r}+b^2e^{5as}(1+2s)+6(-3+b)e^{4r+as}(1+(-1+ab)s)- \\ 3e^{r+4as}\left(\begin{array}{c} (-1+s)^3-2b(-1+s)(1+a(-1+s)s)+ \\ ab^2s(3+a(-1+s)s) \end{array} \right)+ \\ e^{2r+3as}\left(-6(-1+s)^2+6b\left(1-s+s^2\right)+b^2\left(-1-6as^2+6a^2s^2\right)\right)+ \\ e^{3r+2as}\left(\begin{array}{c} 6b(1+a(2-3s))s+3\left(4-7s+3s^2\right)+ \\ b^2\left(-1+(-2+3a)s+9a^2s^2\right) \end{array} \right) \end{array} \right) /$$
$$\left(6b^2\left(e^{2r}+e^{2as}(-1+s)\right)\left(e^{3r}+e^{3as}(-1+s)\right)s^2\right).$$

Function $G(x,y)$ defined in the set $W_l^s(X^*)$ in (6.5)

$$
\begin{aligned}
G(x,y) \;=\;& 3.88963\times 10^{-7}x^{10} + x^9(-7.02174\times 10^{-6} - 0.0000106267y) + \\
& 0.000130647x^8(-0.0592973+y)(1.38083+y) - 0.000951822x^7 \\
& (-0.0727274+y)(-0.0457832+y)(2.10081+y) - 0.0149194x^5 \\
& (-0.015067+y)(0.283461+y)(3.53756+y)(0.147047-0.502117y+y^2) + \\
& 0.00455074x^6(0.165492+y)(2.81952+y)(0.0628253-0.341945y+y^2) + \\
& 0.0339672x^4(0.408308+y)(4.25521+y) \\
& (0.261585-0.668078y+y^2)(0.0115897-0.0308386y+y^2) + \\
& 0.0543289x^2(0.645219+y)(5.68989+y)(0.581562-1.22942y+y^2) \\
& (0.46937-0.520477y+y^2)(0.21724+0.70092y+y^2) - 0.0530287x^3 \\
& (4.97262+y)(0.380765-0.859178y+y^2)(0.148752-0.305255y+y^2) \\
& (0.167076+0.817174y+y^2) - 0.0329843x(-0.944928+y)(1.08797+y) \\
& (6.40704+y)(1.24989-1.67028y+y^2)(1.11048-0.32936y+y^2) \\
& (0.996763+1.39645y+y^2) + 0.00901147(-1.03568+y) \\
& (7.12413+y)(1.69355-2.1394y+y^2)(1.72874-0.827475y+y^2) \\
& (1.45171+1.05599y+y^2)(1.66487+2.43009y+y^2)
\end{aligned}
$$