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Mimeomatroids

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A mimeomatroid is a matroid union of a matroid with itself. We develop several properties of mimeomatroids, including a generalization of Rado's theorem, and prove a weakened version of a matroid conjecture by Rota[2].

Key Words: mimeomatroid, matroid

1. INTRODUCTION

A mimeomatroid is constructed by taking the matroid union of a matroid with itself. This simple operation can be used to find a generalization of Rado's theorem – to address the question of when a family of subsets of a matroid has multiple transversals, each independent in the matroid. It can also partially confirm a matroid conjecture of Rota[2] concerning whether the elements of n bases B_1, B_2, \dots, B_n of a rank- n matroid can always be repartitioned into other bases B'_1, B'_2, \dots, B'_n so that $|B_i \cap B'_j| = 1$ for all i, j .

We begin by recalling several key notions involving matroid union and matroid duality, an intersection theorem of Edmonds, and Rado's theorem. For more background on these topics, see [3, 6].

Let M_1 and M_2 be matroids on ground set E having independent sets I_1 and I_2 , respectively. Then $I = \{I_1 \cup I_2 : I_1 \in I_1, I_2 \in I_2\}$ is the set of independent sets of a matroid on E , $M_1 \vee M_2$. Furthermore, for $X \subseteq E$, the rank of X in $M_1 \vee M_2$ is $\min\{|X \setminus Y| + r_1(Y) + r_2(Y) : Y \subseteq X\}$.

We now recall two classical theorems of matroid theory.

THEOREM 1.1 (Whitney[7]). *Let M be a matroid with ground set E . Let $\mathbf{B}^* = \{E \setminus B : B \text{ is a basis of } M\}$. These are the bases of a matroid*

* Portions of this work comprise part of the author's PhD thesis written under the direction of Richard Brualdi at the University of Wisconsin at Madison.

$M^*(E, \mathbf{B}^*)$, called the dual matroid to M . Let r^* denote the rank function of M^* . Then, for all subsets X of the ground set E , $r^*(X) = |X| - r(E) + r(E - X)$.

THEOREM 1.2 (Edmonds[1]). *Let M_1 and M_2 be matroids with a common ground set E , and rank functions r_1 and r_2 respectively. Then there is a k -element subset of E that is independent in both M_1 and M_2 if and only if, for all subsets X of E , $r_1(X) + r_2(E - X) \geq k$.*

Let $(A_j : j \in J)$ be a family of subsets of a set E . We recall that a transversal of this family is defined to be a set of $|J|$ distinct elements $\{e_1, e_2, \dots, e_{|J|}\} \subseteq E$ with $e_j \in A_j$ for each $j \in J$. We now recall Rado's classical theorem on transversals.

THEOREM 1.3 (Rado[5]). *Let $(A_j : j \in J)$ be a family of subsets of a set E . Let M be a matroid on E having rank function r . Then $(A_j : j \in J)$ has a transversal that is independent in M if and only if, for all $K \subseteq J$, $r(\bigcup_{j \in K} A_j) \geq |K|$.*

In the following section we define mimeomatroids and derive some of their properties. This includes a generalization of Rado's theorem and several relationships between the rank functions of mimeomatroids of different multiplicities. In the final section we use those properties to confirm a weaker version of a conjecture of Rota[2].

2. MIMEOMATROIDS

Let M be a matroid, and let $d \in \mathbf{Z}^{\geq 1}$. Consider the matroid union $\bigvee_{i=1}^d M_i$, where $M_i = M$ for $1 \leq i \leq d$. We call this matroid a *mimeomatroid* of multiplicity d , with rank function $r_d()$, and abbreviate $\bigvee_{i=1}^d M_i$ as $\bigvee^d M$. Observe that if $d_1 \geq d_2$, then $r_{d_1}(X) \geq r_{d_2}(X)$ for any $X \subseteq E$.

The rank functions of mimeomatroids of various multiplicities are related by the following theorem.

THEOREM 2.1. *Let M be a matroid with ground set E , and let $X \subseteq E$. Let $a \geq b \geq c \geq d \geq 1$. Then $\frac{r_a(X)}{a} \leq \frac{r_b(X) + r_c(X)}{b+c} \leq \frac{r_d(X)}{d}$.*

Proof. Let $I_a^1 \dot{\cup} I_a^2 \dot{\cup} \dots \dot{\cup} I_a^a \subseteq X$, $I_b^1 \dot{\cup} I_b^2 \dot{\cup} \dots \dot{\cup} I_b^b \subseteq X$, $I_c^1 \dot{\cup} I_c^2 \dot{\cup} \dots \dot{\cup} I_c^c \subseteq X$, $I_d^1 \dot{\cup} I_d^2 \dot{\cup} \dots \dot{\cup} I_d^d \subseteq X$ each be maximal and independent disjoint unions of independent sets in M . Rearrange superscripts if necessary to have $|I_\alpha^1| \geq |I_\alpha^2| \geq \dots \geq |I_\alpha^a|$ for $\alpha = a, b, c, d$. We have $r_a(X) = |I_a^1 \dot{\cup} I_a^2 \dot{\cup} \dots \dot{\cup} I_a^a| \leq |I_a^1 \dot{\cup} I_a^2 \dot{\cup} \dots \dot{\cup} I_a^d| + (a-d)|I_a^d| \leq r_d(X) + (a-d)|I_a^d| \leq r_d(X) + (a-d)\frac{r_d(X)}{d} = \frac{a}{d}r_d(X)$. Similarly, we have $r_a(X) \leq \frac{a}{b}r_b(X)$, $r_a(X) \leq \frac{a}{c}r_c(X)$, $r_b(X) \leq \frac{b}{d}r_d(X)$, and $r_c(X) \leq \frac{c}{d}r_d(X)$. Combining these inequalities, we get $(\frac{b}{a} + \frac{c}{a})r_a(X) \leq r_b(X) + r_c(X) \leq (\frac{b}{d} + \frac{c}{d})r_d(X)$, from which the theorem follows. \blacksquare

This result is in some sense best possible, since for the matroid $U_{0,n}$ we have $0 = \frac{r_a(X)}{a} = \frac{r_b(X)+r_c(X)}{b+c} = \frac{r_d(X)}{d}$ for any $X \subseteq E$.

The following are several natural corollaries of Theorem 2.1.

COROLLARY 2.1. *Let M be a matroid on ground set E , and let $X \subseteq E$. Then for a mimeomatroid of any multiplicity d , we must have $dr(X) \geq r_d(X) \geq r(X)$.*

COROLLARY 2.2. *Let M be a matroid on ground set E , and let $X \subseteq E$ be independent in $\bigvee^d M$ for some d . Then for any $1 \leq i \leq d$, we must have $r_i(X) + r_{d-i}(X) \geq |X|$.*

The following is a generalization of Rado's Theorem (1.3) to mimeomatroids; Rado's Theorem corresponds to $d = 1$.

THEOREM 2.2. *Let $(A_j : j \in J)$ be a family of subsets of a set E . Let M be a matroid on E . Let $d \in \mathbf{Z}^{\geq 1}$. Then, $(A_j : j \in J)$ has d transversals $\{e_j^i : e_j^i \in A_j, 1 \leq i \leq d, j \in J\}$ independent in the mimeomatroid $\bigvee^d M$ if and only if, for all $K \subseteq J$, $r_d(\bigcup_{j \in K} A_j) \geq d|K|$.*

Proof. First, we assume that $(A_j : j \in J)$ has d transversals $\{e_j^i : e_j^i \in A_j, 1 \leq i \leq d, j \in J\}$ independent in $\bigvee^d M$. Let $K \subseteq J$. Set $X = \{e_j^i : 1 \leq i \leq d, j \in K\}$. By construction, we have $X \subseteq \bigcup_{j \in K} A_j$ and X is independent in $\bigvee^d M$. Hence, we must have $r_d(\bigcup_{j \in K} A_j) \geq r_d(X) \geq |X| = d|K|$.

Now, we assume that for all $K \subseteq J$, $r_d(\bigcup_{j \in K} A_j) \geq d|K|$. For convenience, set $D = \{1, 2, \dots, d\}$. Consider the family of subsets $(A_j^i : i \in D, j \in J)$, with $A_j^i = A_j$.

If we take any $K' \subseteq D \times J$, then we must have $r_d(\bigcup_{(i,j) \in K'} A_j^i) \geq |K'|$.

This is because we can set $K \subseteq J$ minimal so that $K' \subseteq D \times K$, and get $r_d(\bigcup_{(i,j) \in K'} A_j^i) = r_d(\bigcup_{j \in K} A_j) \geq d|K| \geq |K'|$.

We now observe that $\bigvee^d M$ is a matroid on E with rank function r_d , that $(A_j^i : i \in D, j \in J)$ is a family of subsets of E , and that for all $K' \subseteq D \times J$, we have $r_d(\bigcup_{(i,j) \in K'} A_j^i) \geq |K'|$. By Theorem 1.3, there must be a transversal

of $(A_j^i : i \in D, j \in J)$ that is independent in $\bigvee^d M$. This transversal is also d transversals of $(A_j : j \in J)$, from which the theorem follows. ■

Observe that the $\{e_j^i : e_j^i \in A_j, 1 \leq i \leq d, j \in J\}$ provided by the theorem can be partitioned into d sets, each independent in M , whose union is d transversals of $(A_j : j \in J)$. This condition is weaker than having d transversals, each independent in M .

3. APPLICATION

Let M be a matroid of rank n on ground set E . Suppose B_1, B_2, \dots, B_n are pairwise nonintersecting bases of M . Rota conjectured in [2] that there always exists an $n \times n$ matrix A , whose j th column consists of the elements of B_j , ordered in such a way that the rows of A are bases as well.

We now confirm a weaker version of this conjecture, namely that there always exists an $n \times n$ matrix A whose j th column consists of the elements of B_j , and that the first d rows are a disjoint union of d bases, for each $1 \leq d \leq n$. An entirely different approach to this problem using jump systems can be found in [4].

THEOREM 3.1. *Let M be a matroid of rank n on ground set E . Let B_1, B_2, \dots, B_n be pairwise nonintersecting bases of M . There exists an $n \times n$ matrix A whose j th column consists of the elements of B_j and with the first d rows a basis of $\bigvee^d M$, for each $1 \leq d \leq n$.*

Proof. We proceed by induction down from n . The base case of $d = n$ is trivial. We now assume as given $S \subseteq E$, a basis of $\bigvee^{d+1} M$ with $|S \cap B_j| = d + 1$ for $1 \leq j \leq n$. Set $J = \{1, 2, \dots, n\}$. Let $K \subseteq J$. By Theorem 2.1 we have $r_d(\bigcup_{j \in K} B_j \cap S) \geq \frac{d}{d+1} r_{d+1}(\bigcup_{j \in K} B_j \cap S) = \frac{d}{d+1} |\bigcup_{j \in K} B_j \cap S| = \frac{d}{d+1} |K|(d + 1) = d|K|$. By Theorem 1.3, we must therefore have d

transversals $\{e_j^i : e_j^i \in B_j \cap S, 1 \leq i \leq d, j \in J\}$ that are independent in $\bigvee^d M$. If we set $T = \bigcup e_j^i$, then $|T| = dn$ and hence T is a basis of $\bigvee^d M$. ■

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